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CONNECTIVITY FOR RIGIDITY

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Abstract. We characterize the rigidity of the finite part of the parallelogram tiling using some diagonals of the parallelograms as bracing elements. As a consequence of this result we obtain the smallest number of the bracing elements. Our result useful from an algorithmic point of view.

1. Introduction

Frameworks consist of rigid rods and rotatable joints. Consider the framework with white rods on the left hand side in Fig. 2. The framework could deform, as pictured in the Fig. 2 with gray rods. If we wish to prevent this motion and make the framework rigid, we may use diagonal braces, one such a diagonal brace is illustrated in one of the squares in the left hand side in Fig. 2. It makes no difference which of the two diagonals we use in the square.

Definition 1.1. A framework is rigid if any continuous motion of the joints that keeps the length of every rod fixed, also keeps the distance fixed between every pair of joints.

The concept of the rigidity and infinitesimal rigidity are closely related.

The infinitesimal rigidity of a rod-joint framework can be formulated as a rank condition of the rigidity matrix, see [10]. The infinitesimal rigidity implies the rigidity, the converse is not true. Our framework in Fig. 1 is rigid but not infinitesimally rigid, because there is an infinitesimal motion to the direction of the arrows. We can characterize infinitesimal rigidity better than the rigidity of a framework. Generally we have to determine the rank of the rigidity matrix of the framework. But Bolker and Crapo [1] gave a graph theoretical model for square grid framework. There exist some important results in [2],[3],[4],[5],[7],[9], for square grid framework if we use long diagonals, cables or struts, or we allow some "holes" in the grid. To find the rank of the rigidity matrix requires $O\big((lm)^3\big)$ operations for the $l \times m$ square grid framework, but this number decreases to $O(lm)$ the consequence of the graph theoretical model. We give a similar model for parallelogram tiling. It is easy to see that planar square grids with diagonals behave

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very similarly to planar grids of parallelograms which form the same topology. In this paper we solve the general problem of rigidity of tiling of parallelograms.

Consider a rod-joint framework in the form of an $l \times m$ rectangular grid. This framework is not rigid. How can we add diagonal braces to some of the squares as to make a framework rigid? This problem could be solved in generally by a result of Maxwell [6],and a good characterization of grid bracing problem was given by Bolker and Crapo [1]. A similar problem was solved in [4], using long diagonal braces.

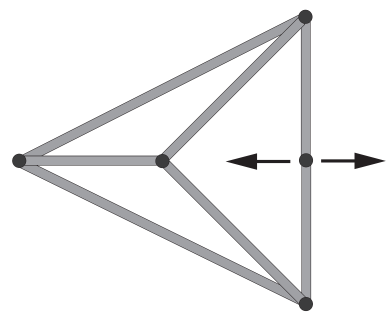

Figure 1

We give a new proof of Bolker–Crapo's theorem. Furthermore we also consider the parallelogram tiling. To make the parallelogram tiling rigid we use bracing elements along one of the diagonals of some parallelograms.

1.1. s-graph

Define the s-graph (skeleton graph) of the framework F as follows: the nodes of the s-graph correspond to the joints of framework F and there is an edge between two nodes of the s-graph if and only if there is a rod between the corresponding two joints of the framework.

Consider a rod-joint framework in one dimension (on a line or on a circular arc with radius r). In case of circular arc, the lengths of the rods less than $2r$ considering the infinitesimal motions.

Lemma 1.2. *A framework is rigid in one dimension if and only if its s-graph is connected.*

Proof. The connectivity of the s-graph means we can get from every node to every other node along edges of the graph. If the s-graph is connected then the joints of the framework can move only together to the same direction. If the s-graph is not connected then the frameworks corresponding to its components can move independently of each other, so the framework is not rigid. □

Consider the usual square tiling (square grid) on the plane with unit edges in which its squares are in rows and in columns. A finite set of those unit square is said to be a square system. A square system is semi-convex if the intersection of the square system and every line that is parallel with the translation vector of the rows or parallel with the translation vector of the columns is a segment. We generally mention $l \times m$ square grid or rectangular grid frameworks, which have exactly l columns and in each column there are exactly m squares. The notion of the semi-convex square system framework is more general than an $l \times m$ square grid framework. We can see a semi-convex square system framework the left hand side in Fig. 2 with white rods.

Let us correspond joints to the vertices of the squares and correspond rods the sides of the squares. Hence we obtain a square system framework. By inserting braces in the diagonals of some squares we want to make the semi-convex square system framework rigid.

The horizontal rods in the column X_i are parallel with each other during any motion of the joints so they can be denoted by vector X_i. Similarly, the vertical rods in the row Y_j are parallel with each other during any motion of the joints so they can be denoted by vector Y_j see(Fig. 2). It is not disturbing if denote the rows and columns by these vectors as well. Thus we can describe the deformation of the square system framework with some vectors disregarding the translation of the framework. These vectors form the so-called descriptive framework. The vectors of the descriptive framework can rotate around the origin independently with respect to each other if there is no diagonal brace in the framework. In Fig. 2, we can see a square system framework that is deformated, because it has not enough number of diagonal braces.

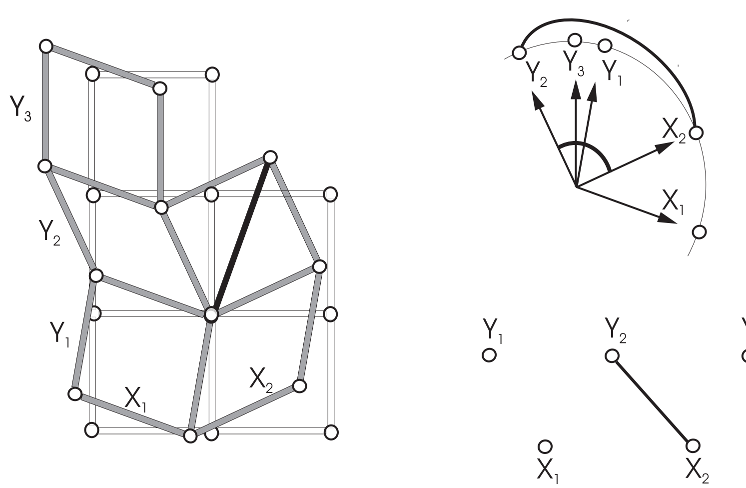

Figure 2

There is only one diagonal brace in column X_2 and in row Y_2. The descriptive framework of this semi-convex square system framework can be seen on the top of the right hand side of Fig. 2.

2. The rigidity of the semi-convex square tilings

We can get to the so-called auxiliary framework of the framework if we correspond joints to the vectors of the descriptive framework. The joint X_i corresponds to vector X_i, the joint Y_j corresponds to vector Y_j and there is a rod between joint X_i and joint Y_j if there is a diagonal brace in the corresponding square in column X_i and row Y_j. The auxiliary framework of the square system framework is on the top of the right hand side of Fig. 2 around the descriptive framework.

The auxiliary graph of the braced square system framework is a bipartite graph. The node X_i in the first node class corresponds to vector X_i, and the node Y_j in the second node class corresponds to vector Y_j, and an edge X_iY_j exists if and only if there is a diagonal brace in the square determined by the column X_i and the row Y_j. In this case vector X_i is perpendicular to vector Y_j in the descriptive framework. It is trivial that the s-graph of the auxiliary framework is isomorphic to the auxiliary graph of the framework. We can see the bracing graph or the auxiliary graph of the framework on the bottom of the right hand side of Fig. 2, below the descriptive framework.

If the auxiliary framework of the square system framework is rigid then every vector X_i is perpendicular to every vector Y_j in the descriptive framework. In this case the square system framework is rigid in the plane.

Theorem 2.1. *A semi-convex square system framework with some diagonal braces is rigid if and only if its auxiliary graph is connected.*

Proof. The joints in the auxiliary framework are on a unit circle and there is some rod between them. The length of these rods is $\frac{\pi}{2}$. The square system framework is rigid in the plane if and only if the auxiliary framework is rigid on the circle. This framework lies on a one dimensional circular arc. Using Lemma 1.2 this framework is rigid if and only if its s-graph is connected. But this s-graph is isomorphic to the auxiliary graph, because their nodes correspond to the columns and to the rows of the square tiling framework, and their edges correspond to the diagonal braces.

If the bracing graph of the square grid framework is not connected, then the square grid framework is not rigid, see the graph in the left hand side in Fig. 2. We can see a result of a possible motion of the joints on the Fig. 2, since X_2 Y_2 is an independent component of the bracing graph. □

This result can be generalized in several directions.

Corollary 2.2. *Theorem 2.1 is true for not degenerated parallelogram tilings (such a tiling consists of parallelograms and its s-graph is isomorphic with the s-graph of a square system framework) instead of square tiling, but it is not true for degenerated parallelogram tilings.*

In [8], Radics and Recski wrote citing W. Whiteley, private communication, June 1990: "it is easy to see that all the results in square grid frameworks are almost the same if we have a planar grid of parallelograms. The only changes will arise when we construct the linear equations or inequalities because the coefficients depend on the size of the parallelograms." In this case the s-graph of the grid of parallelograms is isomorphic with the s-graph of the square system framework.

Corollary 2.3. *We regard the tiling with regular triangles as a non-degenerated tiling with diagonal braces in each parallelogram. Hence the framework of tiling with regular triangles is rigid.*

3. The rigidity of the parallelogram tiling

Now we describe the rigidity of a more general parallelogram system framework when its s-graph is not isomorphic with an s-graph of the square system framework. Define an equivalence relation between the rods. Let us call two opposite rods of a parallelogram equivalent. Hence every rod of the parallelogram system is in one of the equivalence classes.

There is a segment between the middle points of the opposite rods of the parallelogram. A broken line is a maximal line that consists of those segments that are between rods which are in the same equivalence class. Let us take the finite parts of the parallelogram tiling and assume that it is semi-convex, we illustrate them by thin black line. The rods which are in the same equivalence class can move only parallel to each other. If two broken lines that represent different equivalence classes, and the rods of the classes are parallel also, then the broken lines do not intersect each other. If they could intersect each other, then the all sides of parallelogram in the intersection would be parallel, hence this parallelogram would be degenerate.

The motion of a rod is independent from the motion of the other rods that are different equivalence class. This means, if there is no diagonal brace in the framework, then their vectors can circulate around the origin independently from each other. Hence the motion of these rods can be characterized by a vector. Thus we can describe the motion of the parallelogram system framework with some vectors disregarding the congruent transformations of the framework. These vectors form the descriptive framework of the parallelogram system framework. We can see the descriptive framework of a parallelogram system framework on the top of the right hand side of Fig. 3, the broken lines are denoted by thin black line.

Similarly to the square system framework we can construct the auxiliary framework of the parallelogram system framework. The s-graph of the auxiliary framework will be the auxiliary graph of the braced parallelogram system framework. We can see the auxiliary graph of the parallelogram system framework on the bottom of the right hand side of Fig. 3.

Theorem 3.1. *A non degenerate semi-convex parallelogram system framework with some diagonal braces is rigid if and only if its auxiliary graph is connected.*

Proof. Similar to proof of Theorem 2.1. □

The result of Theorem 2.1 and 3.1 is useful from an algorithmic point of view.

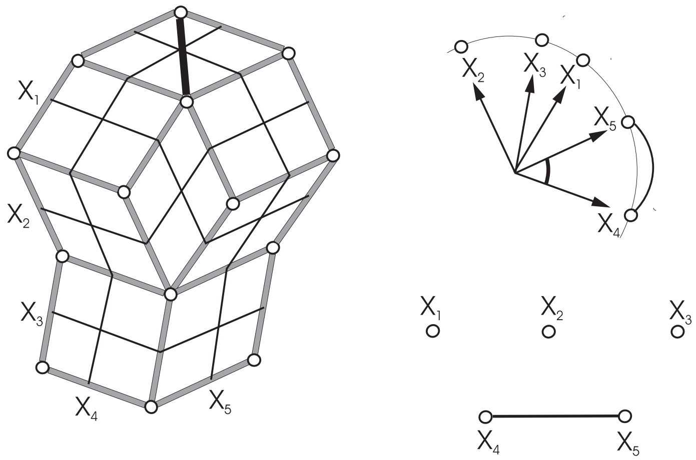

Figure 3

Corollary 3.2. *The number of the joints is $O(lm)$ in case of the $l \times m$ square grid and the size of the auxiliary graph of the framework, and hence the time complexity of the proposed algorithm is $O(lm)$, because the maximal number of the steps of checking connectivity is the maximal number of the edges in the graph, while according to Maxwell the time complexity would be $O\big((lm)^3\big)$, if we use Gaussian elimination for deciding the rank of the rigidity matrix. In case of the parallelogram tiling the size of the auxiliary graph of the framework, and hence the time complexity of the proposed algorithm is $O(r^2)$, where r is the number of the different equivalence classes.*

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