ANNALES UNIV. SCI. BUDAPEST., **49** (2006), 143–154

TESSELLATION-LIKE ROD-JOINT FRAMEWORKS

By

GYULA NAGY

(Received July 13, 2000)

1. Introduction

One of the simplest structures in statics are the tessellation-like rod-joint frameworks including grid frameworks.

Consider a rod-joint framework in the form of a rectangular grid. This framework is not rigid. How can one add diagonal braces to some of the squares in such a way as to make the framework rigid? This problem was solved generally by Maxwell [5], but in his result the time complexity of deciding the rigidity is $O(n^3)$ where n is the number of the joints. We are interested in an algorithm with time complexity $O(n)$ or less. Such a result for grid bracing problem was given by Bolker and Crapo [1, 2] and also by Gáspár, Radics, and Recski [4].

In this paper we give a new proof of Bolker–Crapo's theorem and we consider the eight semiregular (Archimedean) tessellations, and the special hexagon bracing problem in the plane. The word "special" means that we assume the opposite edges of the regular polygon in the tessellation remain parallel during any motion of the vertices. These result are useful from an algorithmic point of view, for instance the number of the joints is $O(n^2)$ in case of the $n \times n$ grid and the size of the auxiliary graph of the framework, and hence the time complexity of the proposed algorithm is $O(n^2)$ while according to Maxwell the time complexity would be $O(n^6)$ if we use Gaussian elimination for deciding the rank of the rigidity matrix.

2. Tessellation-like rod-joint framework

2.1. c-graph

Define the **c**-graph (corresponding graph) of the framework F as follows: the vertices of the **c**-graph correspond to the joints of the framework F and there is an edge between two points of **c** if and only if there is a rod between the corresponding two joints of the framework. Consider a rod-joint framework in one dimension (on a line or on an arc).

LEMMA 2.1. *A framework is rigid in one dimension if and only if its* **c***-graph is connected.*

PROOF. The connectivity of the **c**-graph means we can get from every point to every other point along edges of the graph, that means, the joints of the framework can move together to the same direction with the same velocity. If the **c**-graph is not connected then the frameworks corresponding to its components can move independently of each other. ∎

2.2. Square tessellation

Consider the square tessellation on the plane with unit edges. Its squares are in rows and in columns.

A finite set of unit squares from the tessellation is semi convex if every line parallel with the row or the column intersects the set in an interval.

Let us correspond joints to the vertices of the semi convex tessellation and correspond rods to the sides of the semi convex tessellation. Hence we get to the square tessellation-like rod-joint framework.

Inserting braces in the diagonals of some squares we want to make the square grid rigid. We can see a square tessellation-like rod-joint framework with some braces in Fig. 1. The horizontal rods of the i-th column are parallel with each other during any motion of the joints so they can be denoted by a vector x_i. Similarly, the vertical rods of the j-th row are parallel with each other during any motion of the joints so they can be denoted by a vector y_j.

Thus we can describe the move of the square tessellation-like rod-joint framework with some vectors disregarding the translation of the framework. These vectors form an auxiliary gadget. The vectors of the auxiliary gadget can rotate independently of each other around the origin if there is no diagonal brace in the framework. We can see the auxiliary gadget of the framework of

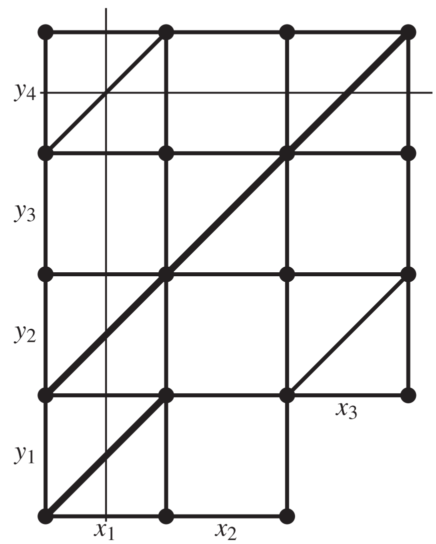

FIG. 1.

Fig. 1 on top of the right hand side in Fig. 2. We translate the vectors to top of the right direction because of the visibility.

The auxiliary graph of the braced square tessellation-like rod-joint framework is a bipartite graph. The point x_i in the first point class corresponds to vector x_i and the point y_j in the second point class corresponds to y_j, and an edge $x_i y_j$ exists if and only if there is a diagonal brace in the square determined by the i-th column and the j-th row. In this case the vector x_i is perpendicular to vector y_j in the auxiliary gadget.

If every square of the square tessellation-like rod-joint framework remains square during any motions of the joints then the square tessellation-like rod-joint framework is rigid in the plane. In this case every vector x_i is perpendicular to every vector y_j in the auxiliary gadget.

THEOREM 2.2. *The square tessellation-like rod-joint framework with some diagonal braces is rigid if and only if its auxiliary graph is connected.*

PROOF. The head of the vectors in the auxiliary gadget are on a unit circle and some of them are of distance $\frac{\pi}{2}$ from each other. These vectors are originally perpendicular in the gadget. Let us construct a framework on the circle. Its joints are the head of the vectors and its rods exist if there is a diagonal brace in the corresponding square, that means these two vectors must be perpendicular to each other. The square tessellation-like rod-joint framework is rigid in the plane if and only if the former framework is rigid on the circle. This framework lies on a one dimensional circular arc. Using Lemma 2.1 this framework is rigid if and only if its **c**-graph is connected. But this **c**-graph is isomorphic to the auxiliary graph, because their points correspond to the columns and to the rows of the square tessellation-like framework, and their edges correspond to the diagonal braces. If the bracing graph of the square grid framework is not connected then the square grid framework is not rigid, see the graph in the bottom of the right hand side on Fig. 2, which is the bracing graph of the square grid framework on Fig. 1.

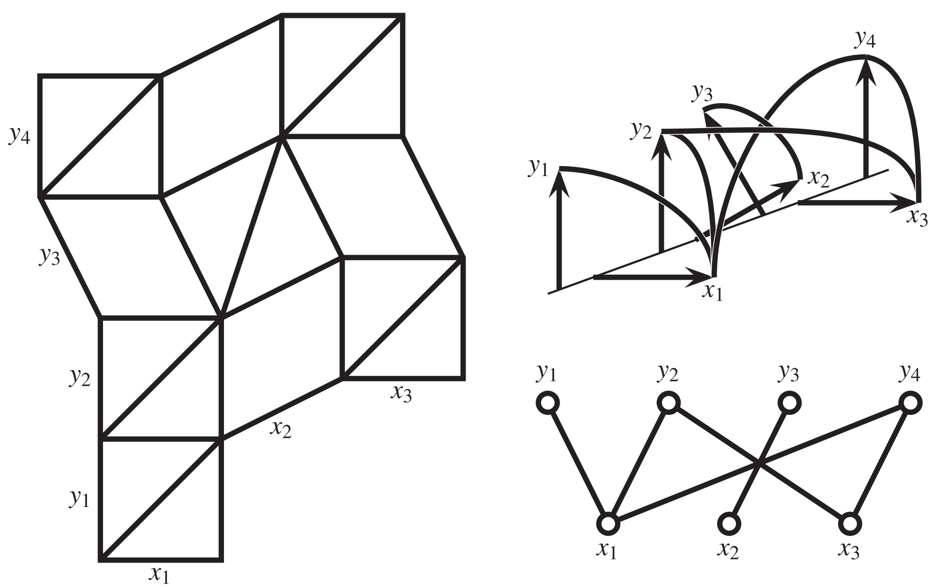

FIG. 2.

We can see a possible motion of the joints on Fig. 2. Since edge x_2y_3 is an independent component of the bracing graph, the corresponding motion of the gadget can be seen at the top of the right hand side of Fig. 2. The motion of the framework (the joints and the rods) is shown on the left hand side of Fig. 2. ∎

This result can be generalized in several directions.

COROLLARY 2.1. *Theorem 2.1 is true for non degenerated parallelogram tessellations (this consists of parallelograms and its* **c** *graph is isomorphic* *to the* **c** *graph of the square tessellation-like rod-joint framework) instead of square tessellation, but not true for degenerated parallelogram tessellations.*

COROLLARY 2.2. *We regard the regular triangle tessellation as a non-degenerated parallelogram tessellation with each diagonal brace. Hence the regular triangle tessellation-like rod-joint framework is rigid.*

The result of Theorem 2.1 is useful from an algorithmic point of view.

COROLLARY 2.3. *The number of the joints is $O(mn)$ in case of the $m \times n$ square tessellation and the size of the auxiliary graph of the framework, and hence the time complexity of the proposed algorithm is $O(mn)$, because the time complexity of checking connectivity is $O(N)$, where N is the number of the vertices and edges in the graph, while according to Maxwell the time complexity would be $O((mn)^3)$.*

3. Hexagon tessellation and semiregular tessellations

While studying the rigidity of the hexagon tessellation Gábor Fejes Tóth suggested to consider the rigidity of all the semiregular (Archimedean) tessellations.

The semiregular (Archimedean) tessellations have incongruent regular polygons and equivalent surrounding of the vertices. Let us denote such a tessellation by a symbol giving the number of sides of the polygons surrounding a vertex in their proper cyclic order. There are eight semiregular (Archimedean) tessellation in the plane as follows (3,12,12), (4,8,8), (4,6,12), (3,4,6,4), (3,3,3,4,4), (3,3,4,3,4), (3,6,3,6), (3,3,3,3,6) [3].

3.1. The semiregular (Archimedean) tessellation-like rod-joint frameworks

The polygons of a semiregular (Archimedean) tessellation are in rows. In case of the square tessellation the columns will also be called rows. Fig. 1 shows four horizontal rows and three vertical rows. Among each set of parallel rows, one will be denoted by thin line in the figure of the tessellation.

A finite set of polygons from the tessellation is "row semi convex" if every line "parallel" with the rows intersects the set of the polygon in an interval.

Consider a “row semi convex” tessellation. Let us correspond joints to the vertices of the polygons and rods to the sides of the polygons. Hence we obtain a semiregular (Archimedean) tessellation-like rod-joint framework.

3.2. The (3,3,3,3,6)-, (3,3,3,4,4)- and the (3,3,4,3,4) tessellation-like rod-joint frameworks

3.2.1. (3,3,3,3,6) tessellation

Firstly we shall consider the (3,3,3,3,6) tessellation with unit edges.

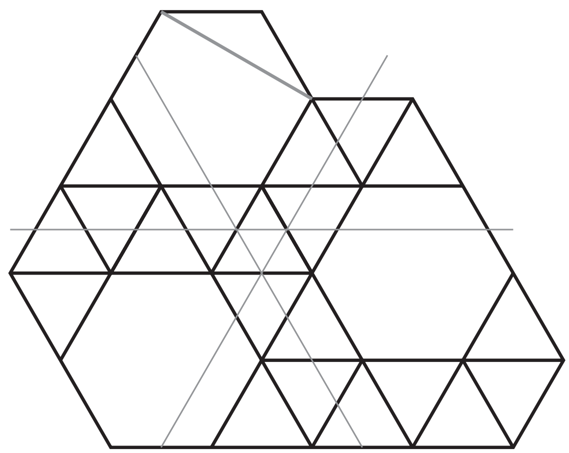

FIG. 3.

It is easy to see that the (3,3,3,3,6) tessellation-like rod-joint framework is almost rigid, because there are many rigid triangles between the polygons of the tessellation (Fig. 3). Some hexagons on the boundary of the framework could be not rigid. These hexagons must be made rigid with short diagonal braces.

3.2.2. (3,3,3,4,4)- and the (3,3,4,3,4) tessellation

The rigidity of the (3,3,3,4,4)- and the (3,3,4,3,4) tessellation-like rod-joint frameworks is implied by Corollary 2.1. In Figures 4 and 5 we denote the rows by thin lines.

We can see the auxiliary gadget at the top of the right and the auxiliary graph at the bottom of the right. In the auxiliary graph some edges are

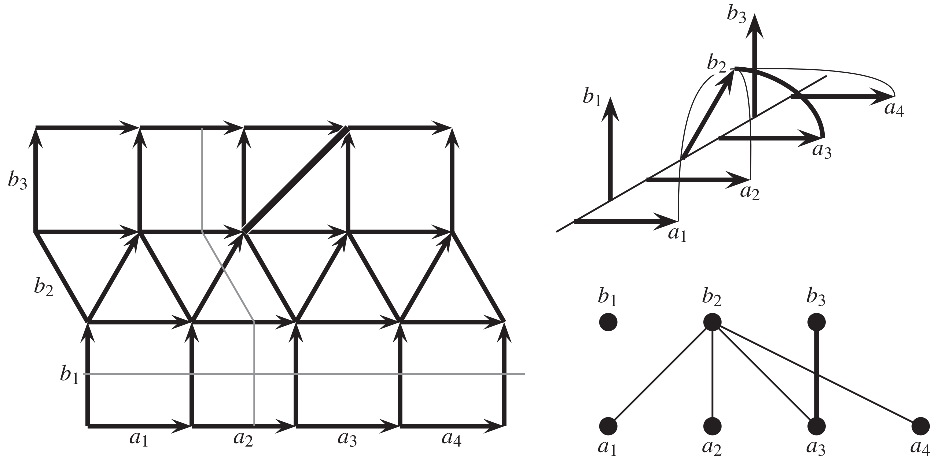

FIG. 4.

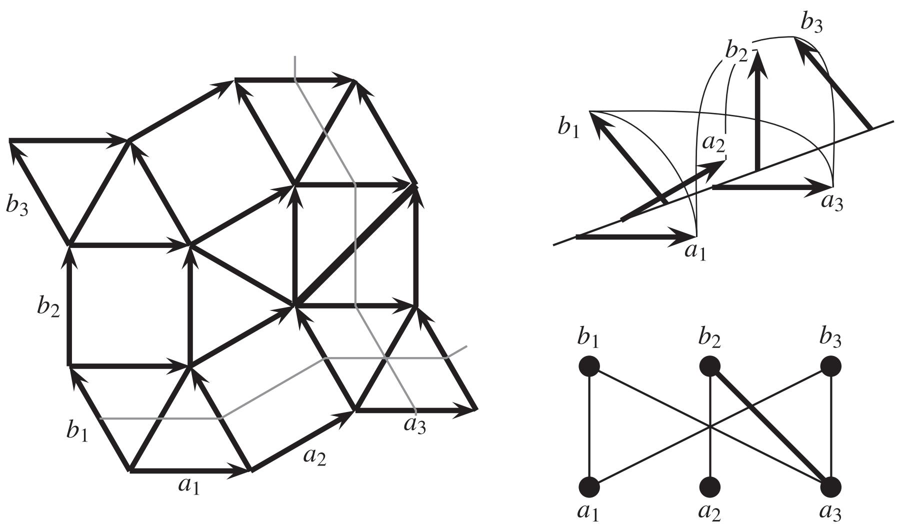

FIG. 5.

originally included because the common edges of the neighboring triangles form diagonal braces in the parallelogram tessellation.

In the auxiliary graphs there will be new edges (denoted by thick line) if we put some diagonal braces into the square of the (3,3,3,4,4)- and the (3,3,4,3,4) tessellation-like rod-joint framework.

THEOREM 3.1. *The (3,3,3,4,4)- and the (3,3,4,3,4) tessellation-like rod-joint frameworks are rigid if and only if their auxiliary graphs are connected.*

PROOF. The statement of the theorem is implied by Corollary 2.1. ∎

COROLLARY 3.1. *In case of the (3,3,3,4,4) framework we have to insert into every square row one diagonal brace, this result is obvious. In case of the (3,3,4,3,4) framework only one diagonal brace is enough to connect the auxiliary graph.*

3.3. The special tessellation frameworks

3.3.1. The special (4,6,12) framework

Let us consider the rest of the special semiregular (Archimedean) tessellation, and the special hexagon tessellation bracing problem in the plane. The word "special" means that we assume the opposite edges of the regular polygon in the tessellation remain parallel during any motion of the vertices. The special assumption is unnecessary in the former cases, but it is necessary for the rest of the special semiregular tessellations.

We can construct this kind of "special" framework using some new rods or joints. In case of the hexagon we add a new joint into the center of the hexagon, and three new rods from the center joint to every second one of the hexagon joints.

To make these tessellations rigid we use bracing elements along the shortest diagonals of the polygons because the two neighboring sides and the shortest diagonals form triangles. Hence the two neighboring sides cannot rotate around the common joint.

The rigidity problems of these tessellations are similar to each other, hence for simplicity we shall restrict our consideration to one of the former frameworks, for example the (4,6,12) framework. The drawing of the framework and auxiliary graph of the (4,6,12) tessellation framework are shown in Fig. 6.

Given a (4,6,12) tessellation with unit edges. Its polygons are in parallel rows that are in six kinds of different direction. The six directions are denoted by thin lines.

The special assumption implies the next statement. The rods of a row that are perpendicular to the direction of the row are parallel with each other during any motion of the joints so they can be denoted by a vector. Thus

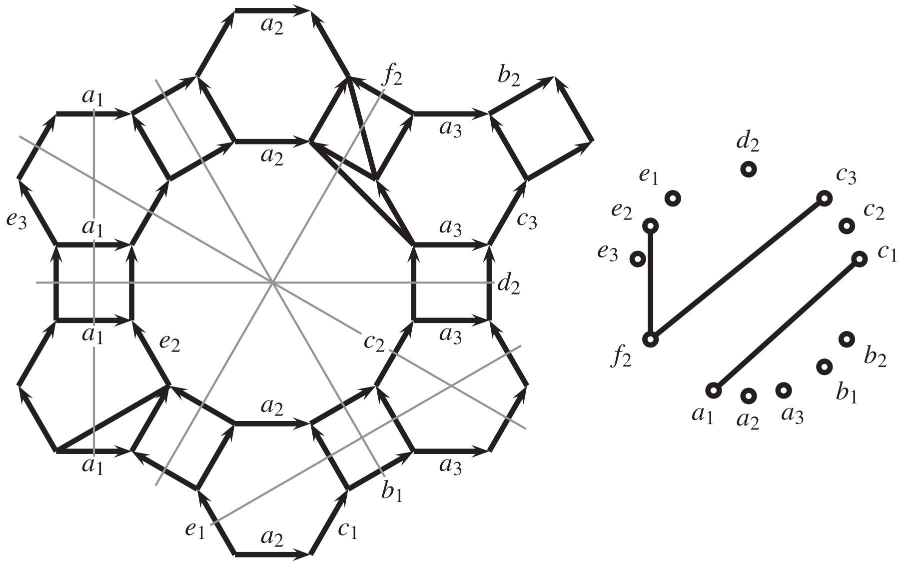

FIG. 6.

we can describe the move of the special (4,6,12) tessellation-like rod-joint framework with some vectors disregarding the translation of the framework only. These vectors form an auxiliary gadget. The vectors of the auxiliary gadget can rotate independently each other around the origin if there is no diagonal brace in the framework. The auxiliary graph of the braced special (4,6,12) tessellation-like rod-joint framework is a sixpartite graph. The point a_i in the first point class corresponds to vector a_i and the point b_j in the second point class corresponds to b_j, and an edge $a_i\ b_j$ exists if and only if there is a short diagonal brace between the a_i-th row and b_j-th row. In this case the head of vector a_i is braced to vector b_j in the auxiliary gadget. The auxiliary gadget is rigid if and only if the special (4,6,12) tessellation-like rod-joint framework is rigid in the plane.

THEOREM 3.2. *The special regular hexagon-, the (4,8,8)- and the (4,6,12) tessellation-like rod-joint framework with some diagonal braces is rigid if and only if its auxiliary graph is connected.*

PROOF. This proof is similar to that of Theorem 1.1. The head of the vectors in the auxiliary gadget are on a unit circle. Let a framework be on this circle. Its joints are the heads of the vectors and its rods exist if there is a diagonal brace in the corresponding rows, that means these two vectors can move together. The (4,6,12) tessellation-like rod-joint framework is rigid in the plane if and only if the former framework is rigid on the circle. This framework lies on a one dimensional circular arc. Using Lemma 1.1 this framework is rigid if and only if its **c**-graph is connected. But the **c**-graph is isomorphic to the auxiliary graph of the framework, because their points correspond to the rows of the framework, and their edges correspond to the diagonal bracing. If the bracing graph of the square grid framework is not connected then the square grid framework is not rigid. ∎

3.3.2. The (3,6,3,6), (3,12,12) and (3,4,6,4) special framework

Let us consider the (3,6,3,6), (3,12,12) and (3,4,6,4) special semiregular (Archimedean) tessellation bracing problem in the plane.

The rigidity problems of these tessellations are similar to each other, hence for simplicity we shall restrict our consideration to one of the former frameworks for instance the (3,4,6,4) framework.

The illustration of the framework, auxiliary gadget and auxiliary graph of tessellation (3,4,6,4) framework are shown on Fig. 7. The three different directions of the row are denoted by thin lines.

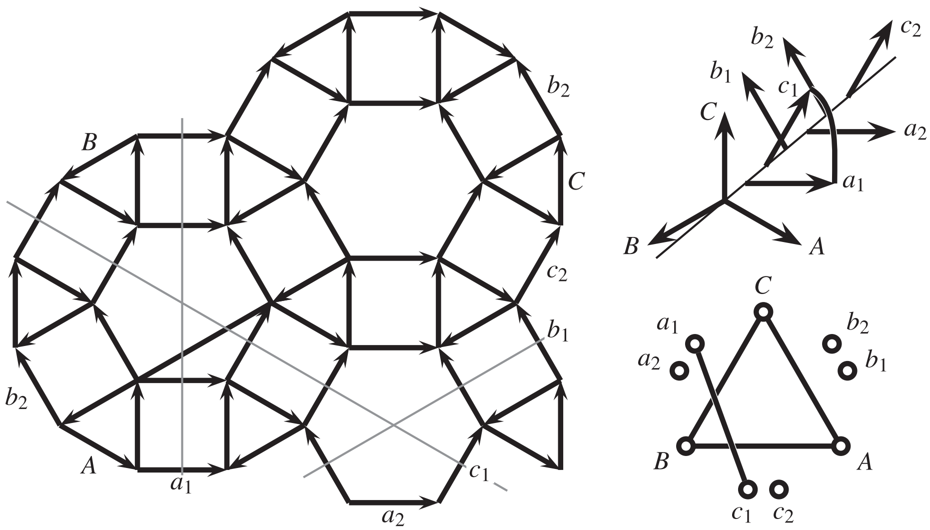

FIG. 7.

We can describe the move of the special (3,4,6,4) tessellation-like rod-joint framework with some vectors disregarding the translation of the framework only. These vectors form the auxiliary gadget of the framework. The vectors of the squares and the hexagons can rotate independently each other around the origin if there is no diagonal brace in the framework, but the vectors of the triangles denoted by capital letter can move together only, because they form a rigid triangle. The auxiliary graph is a sixpartite graph. The point a_i in the first point class corresponds to vector a_i and the points b_j, c_k in the second and the third point classes correspond to vector b_j and c_k respectively, and an edge $a_i\ b_j$ exists if and only if there is a short diagonal brace between the a_i-th row and the b_j-th row. In this case the vector a_i is braced to vector b_j in the auxiliary gadget. The vectors of the triangles denoted by A, B and C are braced with each other, hence between the corresponding points of them there are edges in the auxiliary graph. The auxiliary gadget is rigid if and only if the special (3,4,6,4) tessellation-like rod-joint framework is rigid in the plane.

THEOREM 3.3. *The special (3,6,3,6), (3,12,12) and (3,4,6,4) tessellation-like rod-joint framework with some diagonal braces is rigid if and only if its auxiliary graph is connected.*

PROOF. The proof is similar to that of Theorem 3.2. ∎

COROLLARY 3.2. *Theorems 3.1, 3.2 and 3.3 are true for non degenerated special affine regular hexagon tessellation-like rod-joint frameworks and the affine semiregular tessellation-like rod-joint frameworks respectively (these consist of affine regular polygons and their* **c** *graphs are isomorphic to the* **c** *graphs of the special regular hexagon tessellation-like rod-joint frameworks and the semiregular tessellation-like rod-joint frameworks respectively).*

The result of Theorems 3.1, 3.2, 3.3 are useful from an algorithmic point of view:

COROLLARY 3.3. *Let n be the sum of the number of the rows in the different directions. Hence n equals the number of the points of the auxiliary graph of the special tessellation-like rod-joint framework. The time complexity of the rigidity algorithm of the former framework braced with diagonals is linear in the size of the graph, hence $O(n^2)$, because the time complexity of checking connectivity is $O(N)$, where N is the number of the vertices and edges in the graph.*

Acknowledgment

I received much help from András Recski. This research was supported by the National Science Foundation (grant number: OTKA T 30122).

References

[1] E. D. BOLKER: Bracing grids of cubes, *Env. Plan. B* **4** (1977), 157–172.

[2] E. D. BOLKER and H. CRAPO: Bracing rectangular frameworks I, *SIAM J. Appl. Math.* **36(3)** (1979), 473–490.

[3] L. FEJES TÓTH: *Regular Figures*, Pergamon Press, Oxford (1964).

[4] ZS. GÁSPÁR, N. RADICS and A. RECSKI: Square Grids with Long "Diagonals", *Optimizations Methods and Software* **10** (1998), 217–231.

[5] J. C. MAXWELL: On Reciprocal Figures and Diagrams of Forces, *Phil. Magasine 4* **27** (1864), 250–261.

[6] GY. NAGY: Diagonal bracing of special cube grids, *Acta Technica Acad. Sci. Hung.* **106(3–4)** (1994), 265–273.

[7] N. RADICS: Rigidity of t-story Buildings, in: *Proc. 1st Japanese–Hungarian Symposium on Discrete Mathematics and Its Applications* (1999), 181–187.

[8] A. RECSKI: Bracing Cubic Grids – A Necessary Condition, *Discrete Mathematics* **73** (1988/89), 199–206.

[9] A. RECSKI: *Matroid Theory and its Applications in Electric Network Theory and in Statics*, Akadémiai Kiadó, Budapest and Springer, Berlin (1989).

Gyula Nagy

Szent István University
Ybl Miklós School
Budapest
Hungary
nagygyu@solaris.ymmf.hu