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THE RIGIDITY OF SPECIAL d CUBE GRIDS

By

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1. Introduction

One of the simplest structures in statics are the frameworks.

DEFINITION 1. A framework consists of rigid rods connected by rotatable joints.

DEFINITION 2. A framework is rigid if any continuous motion of the joints that keeps the length of every rod fixed, also keeps fixed the distance between every pair of vertices in the framework.

Let us consider an $n_1 \times n_2 \times \ldots \times n_d$ cube grid in the d space. The corresponding rod and joint framework is a mechanism in the d space. Let the length of the rods be unit. This cube grid framework consists of $(n_1 + +1)\ldots(n_{i-1}+1)n_i(n_{i+1}+1)\ldots(n_d+1)$ pieces of parallel V_i rods, V_i denotes the i-th axe of the d dimensional coordinate system, for $1 \leq i \leq d$. There are $(n_1+1)(n_2+1)\ldots(n_d+1)$ pieces of rotatable joints in the grid.

We suppose that each cube is a rhomboid during any motion of the vertices (thus we disregard those motions of the cube where the vertices of any 'square' face do not remain coplanar). (Throughout, quotation marks will refer to the original situation). Thus the 2^{d-1} pieces of 'parallel' edges of the unit cubes are parallel during the motion of the vertices. If the distance of the two middle points of the opposite edges of each two dimensional square face are unit then the special assumption is satisfied obviously. For instance, this special assumption has to be realized technically by medianrods joining with joints the middle points of the two opposite rods of each two dimensional square face (*Fig. 1.*). This construction allows infinitesimal motion, but it is rigid according to the Definition 2. Naturally, there is not necessary to use the

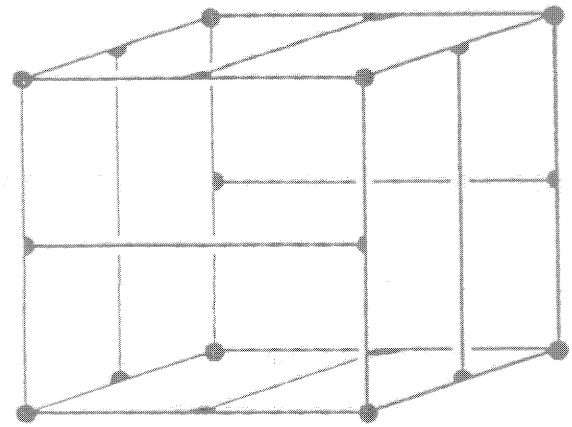

Fig. 1

medianrods in every two dimensional square face, although its exact number and places are interesting and open problem.

BOLKER and CRAPO solved the case $d=2$, when the special assumption is not needed. We want to make the special cube grid rigid using some diagonals along the square faces of the unit cubes as diagonal bracing. The consequence of our special assumption is that the rods between opposite hyperfaces of a cube are parallel to each other. Thus every rod between two 'hyperplanes' are parallel. Naturally these 'parallel hyperplanes' are 'perpendicular' to the V_i. Consider those rods that are 'parallel' to V_i. There are $(n_1+1)\dots(n_{i-1}+ +1)(n_{i+1}+1)\dots(n_d+1)$ pieces of parallel rods between the j-th and $(j+1)$-th neighbour hyperplanes, $1 \le j \le n_i$. These rods are characterized by the vector v_{ij}, $1 \le j \le n_i$, (*Fig. 2.*).

Every square of the grid is characterized by two pieces of former vectors v_{ij}, v_{kl} $(1 \le k \le d,\ 1 \le l \le n_k,\ i \ne k)$. If there is a diagonal brace in one of the characterized squares then the two vectors are perpendicular during any motion of the vertices. Thus applying a diagonal brace for any of the characterized square will fix the others as well.

The special cube grid is rigid if and only if every 'square' remains square during any motion of the joints. Let us define the bracing graph of the special cube grid framework. We have d point classes: V_i, $1 \le i \le d$, and there are n_i points in the point class V_i. The points are also denoted by v_{ij}, because they correspond to the vectors v_{ij}. Let two vertices v_{ij}, v_{kl} $(1 \le k \le d)$, $1 \le l \le n_k$, $i \ne k)$ be adjacent if and only if there is a diagonal brace in the two dimensional square characterized by vectors v_{ij}, v_{kl}.

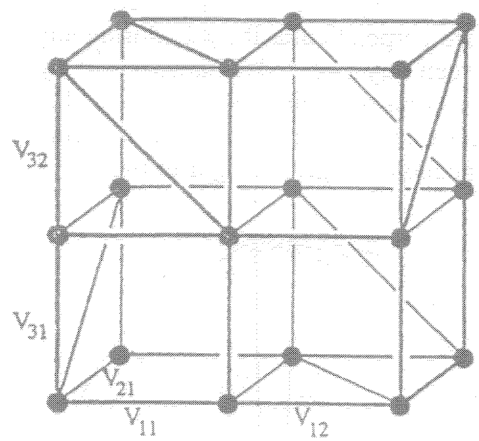

Fig. 2

Thus we have a d-partite graph D. We indicate the diagonal braces as edges in the d-partite graph. If D is complete, that is, if there exists an edge between every pair of points if they are in different point classes, then the cube grid is rigid, because each rhomboid is a cube in the grid.

However, less diagonal bracing may also be sufficient. Consider the $\binom{d}{2}$ pieces of bipartite subgraphs of the d-partite bracing graph D.

2. Necessity

THEOREM 1. *If the special d cube grid bracing with diagonal braces is rigid then the bipartite subgraphs of the d-partite bracing graph are connected.*

PROOF. If the bipartite subgraph V_i V_k is disconnected then there exists a motion which can be composed from elementary motions parallel to the coordinate axes V_i and V_k, as a consequence of the following theorem of Bolker and Crapo: An $n \times m$ square grid bracing with diagonal braces or without diagonal brace is rigid if and only if its bracing graph is connected.

The necessary condition is also sufficient:

THEOREM 2. *The special d cube grid as a bar and joint framework bracing with diagonal braces or without diagonal brace is rigid if and only if the bipartite subgraphs of the d-partite bracing graph are connected.*

Before the proof of the sufficiency we introduce a special framework on the d dimensional sphere. V_{ij} are the end points of the respective grid vectors on the unit sphere. The special cube grid is rigid if and only if every two vectors v_{ij} and v_{kl} are perpendicular if $i \neq k$, during any motion. If there is a diagonal brace between any two rods characterized by vectors v_{ij}, v_{kl} then we denoted it by a rod on the unit sphere between the points V_{ij}, V_{kl}. Thus a framework s arises on the sphere (*Fig. 3.*) If this framework s is rigid on the sphere then the special cube grid is rigid in the space.

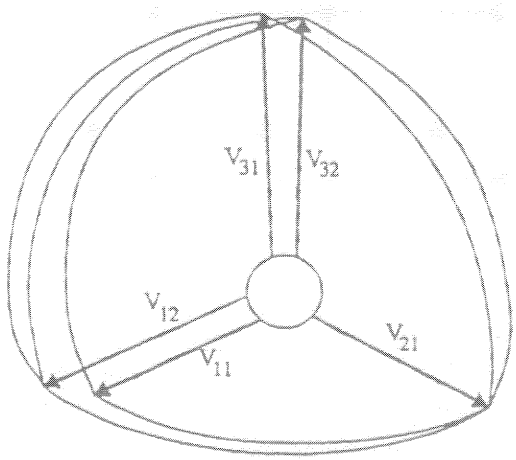

Fig. 3

Define the graph c of framework F as follows: the vertices of the graph c correspond to the joints of F and there is an edge between two points of c if there is a rod between the corresponding two joints of the framework.

We have a framework on the sphere and its graph c is d-partite and its bipartite subgraphs are connected. It is clear from the definitions of the graph c of framework s and the bracing graph D that the graph c of s and the d-partite bracing graph D are isomorphic. D denotes the bracing graph of the original framework and c denotes the graph of the fictitious framework on the unit sphere.

3. Sufficiency

If we can prove that the framework s is rigid on the sphere the Theorem 2 is true. In this case we need the coordinates of the points V_{ij}. Let us introduce a new system of coordinates V_i', where the origin of the coordinate system is still at the centre of unit sphere and let the hyperplane determined by the points V_{ij} be perpendicular to one of the new axes, for example V'_d. The rank of the rigidity matrix [7] depends on the new V'_i coordinates of the points V_{ij} and on the graph c only, since these points form a regular $d-1$ dimension simplex which is parallel to the new hyperplane V'_i, $1 \leq i \leq d-1$. The d-th coordinates of the V_{ij} are equal to each other. Thus we can simplify the original problem.

Consider a special cube grid bracing with some diagonal braces along square faces; it is rigid if and only if framework p is rigid in the former hyperplane. The joints of p are the same as those of framework s and there is a rod between two joints if there is rod in framework s. Since joints of s are in a hyperplane at the beginning of the motion, it suffices to consider the rigidity of the framework p in the $d-1$ dimensional hyperplane.

Let us introduce stil another system of coordinates V''_i, where the origin of the coordinate system is at V'_{dj} and let the hyperplane determined by the points V_{ij}, $1 \leq i \leq d-1$ be perpendicular to one of the new axes, for example V''_{d-1}. The rank of the rigidity matrix depends on the new V''_i coordinates of the points V_{ij} and on the graph c only, since these points form a regular $d-2$ dimensional simplex which is parallel to the new hyperplane V''_i, $1 \leq i \leq d-2$. The $d-1$-th coordinate of the V_{ij}'s are equal to each other. Thus the rank of the rigidity matrix decreases at least by $n_1 + n_2 + \ldots + n_{d-1} + (d-1)(n_d - 1)$ if we delete the joints V_{dj} and the incident rods of framework p.

Repeating the former idea several times we get to the joints V_{1j}, V_{2j} and their framework which is rigid in dimension 1. This completes the proof of the theorem.

In the proof we gave a new framework (framework p) in a hyperplane that is rigid in its plane if and only if the special cube grid is rigid in the space and the graph c of the framework p is isomorphic to the bracing graph of the special cube grid.

Thus we need $(d-1)(n_1 + n_2 + \ldots + n_d) - \binom{d}{2}$ face diagonal braces for the rigidity of an $n_1 \times n_2 \times \ldots \times n_d$ special cube grid.

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