Rigidity and Safety Optimization of 3-Dimensional Frame Systems as Braced Scaffolding with Graphs

Gyula Nagy Kem

Szent István University Ybl Miklós Faculty of Architecture and Civil Engineering Thököly út 74, HU 1146, Budapest, Hungary

E-mail: nagy.gyula@ybl.szie.hu

Abstract: Fixed joints attach the three-dimensional cubic grid structure as Scaffolding its base face and its lateral face to the planar ground and a planar lateral wall, respectively. Inserting some short or long bracing elements, we make the framework of the scaffolding rigid. We provide a convenient method to determine if the braced structure is rigid or flexible. This result is also useful from the algorithmic point of view, and it is applicable in structural and safety engineering, may useful input of simulation and topology optimization. The presented results provide algorithms which contribute to the development of a better characterization of the structures which are discretized as Bar and Joint Scaffolding frames or can be subdivided to those . Last, we suggest the revision of the sample scaffolding one of the standard.

Keywords: Structure optimization, Braced scaffolding, Graph connectivity, Safety of the structure

1 Introduction

We consider the rigidity of the grid, like bar and joint framework that are necessarily finite structure. Building frameworks, crystalline materials, some biomechanical or some social networks with a lattice-like structure, are considered central force model. In this model, the angles between two adjacent bonds can change with zero energy, but then the amount of energy used is not zero when the lengths of the bonds are changed. Hence, this model is more complicated than our bar joint model, which uses regular length bars instead of bonds with the variable lengths. The central-force percolation rigidity model considers the rigidity of the structure when bonds are randomly chosen [1-4]. Significant results consider the rigidity of finite structures [5-17] some finite and some infinite periodic structures. In this paper, we consider frameworks, which consists of fixed length, not deformed bars that could rotate independently and freely around the connecting joint, our frameworks are grid-like finite structures. A natural question for the bar-joint structures is; what is the minimum number of bars and which joints have constrained by bars that will lead to a rigid structure. The framework, consisting of rotatable joints connected by bars, is one of the most simple structures in statics.

Definition 1.1: A framework is rigid if any continuous motion of the joints that keeps the length of every bar fixed also keeps the distance between every pair of vertices fixed in the framework.

Structure, which forms a triangle with vertices as joints, and bars as sides, is rigid, in two and three dimensions also. A similar square frame is not rigid on the plane, and neither is in the space, it could deform to a rhomb in the 2-dimensional plane, and a rhomb or a skew quadrilateral in the 3-dimensional space. If we use one of the diagonals of the square, as bracing elements, then the square framework with this will be rigid in the 2-dimensional plane but will be not rigid in space, it could deform to a skew quadrilateral in the space. If we use both diagonals of a square, as a bracing element, the square with the diagonals will be rigid in the plane according to the above definition. In statics, rigidity does not even allow infinitesimal motions. A framework is infinitesimally rigid if it is rigid, and does not have any infinitesimal motion, i.e. kinematically determinate. For example, the square with the two diagonal braces is not infinitesimally rigid in space. We could fix three joint of the square; in this case, the fourth can move infinitesimally perpendicularly to the plane of the fixed triangle. Hence this framework is referred to kinematically indeterminate in 3-dimensional space (we disregard that the diagonals intersect each other, one of them can move independently from the other).

1.1 The infinitesimal rigidity of bar-joint framework

The infinitesimal motion is a special case of virtual motion; that refers to a virtual change in the position such that the constraints remain satisfied (possible motions). The rigid body motions refer to the trivial infinitesimal motion. Definition 1 allows frameworks that have infinitesimal motions. In the statics for the rigid structure, the non-trivial infinitesimal motions of the joints have not permitted [18-22].

If a framework is infinitesimally rigid, we require first-order preservation of distances during the infinitesimal motions of all joints. We have to decide the rank of rigidity matrix of the framework.

Let (x_i,y_i,z_i) be the coordinates of the joint P_i of the F(P) bar-joint structure, where $(1 \leq i \leq n)$. A bar between the joints P_i and P_j determines the distance from P_i to P_j, therefor it is constant, by differentiating its square, leads to the next equation:

$$(x_i - x_j)(\dot{x}_i - \dot{x}_j) + (y_i - y_j)(\dot{y}_i - \dot{y}_j) + (z_i - z_j)(\dot{z}_i - \dot{z}_j) = 0 \quad (1).$$

Where, the velocity coordinates $\dot{x}_i$, $\dot{y}_i$, $\dot{z}_i$ are the varieties. Hence, if we use bars between joints, and the number of bars is e, then we get a system with e pieces of equations. The matrix representation of the equation system is the next:

$$Au = 0 \qquad (2).$$

Where u is the column vector of velocity, and A is an e×3n rigidity matrix. In statics, rigidity does not even allow infinitesimal motions. In this case, the equation (2) has the trivial solution only (i.e., the rigid body like motions). The rigid body motion of the joint keeps fixed the distance between the pairs of the joint. If the joints of the framework have motions, that different from the rigid body motion, then the framework is not rigid. In this case the *rank(A)<3n-6*. The framework is rigid if and only if the *rank(A)=3n-6*, see: [20, 21]. Maxwell [18] gave this characterization but with his result, the time complexity of deciding the rigidity is $O(e^3)$.

A three-dimensional framework composed of ideal rigid bars connected by ideal frictionless joints is statically and kinematically determinate if the above rank condition is satisfied. The equilibrium equation system describes the internal tensions and loads in the bars and at the nodes. The transpose of the matrix of the equilibrium equation system we get the rigidity matrix. Hence, they have equal rank. We consider the kinematically determinate, i.e. the infinitesimally rigid, and the kinematically indeterminate, i.e. infinitesimally flexible frameworks [20-22]. From this point, the kinematically determined, i.e. the infinitesimally rigid, and the kinematically indeterminate, i.e. infinitesimally flexible refers to motion and rigidity respectively.

The coefficient matrix of system ***A*** is referred to as rigidity matrix of *F(**P**)*. We determine the infinitesimal rigidity of a bar-joint framework after Maxwell as a rank condition of the rigidity matrix. We can see an equation system that describes the possible motion of a cubic framework on Figure1.

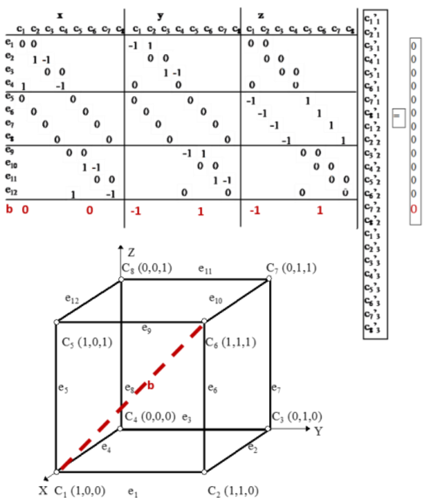

Figure 1

The rigidity equation system of the Bar-Joint framework.

On the bottom, we can see a cubic bar-joint framework with one diagonal brace. It is not a special framework since for instant the joint on the plane YZ are not necessarily coplanar during the motions of the frameworks. We have to solve the above equations system for determining the possible motions.

We know the number of equations are not enough, one of the reasons are is that there are few bars in the framework, the other reason is the rigid body like motions. We can find exact descriptions of the infinitesimal rigidity in [20-22]. In some special cases, we can work with graph or matroid theoretical models from which very fast and efficient algorithms can be obtained [20, 21, 23-28].

The rigid body type motions of the framework are named as trivial infinitesimal motions.

Definition 1.2: An *F(p)* bar-joint framework is infinitesimally rigid, i.e. kinematically determinate if it only has the trivial infinitesimal motions.

A framework is infinitesimally flexible if the above type equation system has more than a 6-dimensional subspace of solutions since the rigid body like motions composes a 6-dimensional subspace. Rigidity does not even allow infinitesimal motions inside our framework, for example, a square its two diagonals is rigid in space but not infinitesimally rigid. Fix the three joints the fourth joint can move infinitesimally perpendicular to the plane of the square (we disregard that the diagonal elements are intersecting each other).

Very few results are known for infinitesimal rigidity in the space. Maxwell [18] solved the most useful result for the general case when the structures of the joints are not grid-like. Using this result, we have to determine the rank of the rigidity matrix of the framework [20-22]. Hence, the order of the time complexity of deciding the rigidity in the 3-dimensional space is N^3, if we use Gauss elimination, where N is the number of the joints. How can we decide in a large framework which bars are redundant and which are not? There are some results in the plane [23-26, 29-36], and space [23, 24, 26-28, 31-32, 37-42] for grid type framework that can decide the infinitesimal rigidity of the framework more simply. The mentioned results are better from an algorithmic point of view, because of the size of the bracing graph of the framework, and hence the time complexity of the proposed algorithm is smaller than consider the rank of the rigidity matrix. Hence, we can use very fast graph theoretical, or combinatorial algorithms for rigidity. We can find some other results that use other technics for the optimization [43-54].

1.2 The rigidity of the periodic structure

Consider a bar-joint framework in the form of a $l \times n$ rectangular grid. How can we add diagonal braces of the grid framework in a way to make the framework rigid? Important results consider the rigidity of finite periodic structures [25, 37, 38, 39, 40] and some infinite periodic structures [43, 44, 49]. We consider frameworks with the tools of combinatorial optimization in this paper. The framework consists of deterministic, fixed length bars that could rotate independently around the connecting joint, our frameworks are grid-like finite structures. A natural question for the finite and lattice-like structures is; what is the minimum number of additional braces and where they should be placed, that will lead to a rigid structure [23-25, 29-33, 35-40]. Hence, we get optimal structures.

Since the joints are the vertices of the grid, the rigidity of the structure could be described in a simpler way. In a similar case, Bolker and Crapo solved the square grid-bracing problem in [4]. The authors in [21] solved the same problem with long bracing. We could regard our results than the generalization of them results in the three-dimensional space.

2 Methods

2.1 How to brace the Cubic grid

Consider a l×m×n rectangular cubic grid. There are ball joints in the grid points, and there are rigid bars in the grid edges. Let the joints Ji;j;k of the cubic grid framework be the points $P_{i,j,k}$ =(i,j,k) of the coordinate system, where i, j and k are integer and $0 \leq i \leq l$, $0 \leq j \leq m$, $0 \leq k \leq n$. There is a bar between two joints if and only if the difference of one of their coordinates is one, and the others are equal. Thus, we have a cubic grid framework. The lengths of the bars are unity. This framework is not rigid. How can we add diagonal braces between two joints of the grid in a way to make the framework rigid? This problem could be solved if we consider the infinitesimal rigidity of the framework, deciding the rank of the rigidity matrix. Using the result of Maxwell, the number of the steps of determining the rigidity is $O(N^3)$, where N is the number of the joints. We are looking for a better characterization that gives a method, which is convenient enough to determine the rigidity for our frameworks. There is no similar result for the cubic grid, with similar characterization to the result of Bolker and Crapo for the square grid in the plane. Bolker described the minimal redundant set of diagonal braces in the cubic grid [24]. Recski [37] gave a necessary condition. The one-story building is a l×m×1cube grid framework whose joints of the bottom l×m face are attached to fixed joints on the ground face. Bolker and Crapo [23] have a result for the short diagonal bracing problem of the one-story building. Radics [40] generalized this result to the case of a n-story building with short diagonal braces.

2.2 Scaffolding with bracing items

2.2.1 The Bar-Joint Scaffolding

For the sake of accuracy, our redefined structures are denoted the bar-joint prefix if it is different from the general engineering practice.

Definition 2.1 The Bar-Joint Scaffolding is a $l \times m \times n$ cube grid framework and its joints on the base $l \times m$ face are attached to fixed joints in the planar ground, which is the *X-Y* plane of the coordinate system. The joints on the back lateral $m \times n$ face also are connected to fixed joints in the lateral wall that is the *Y-Z* plane of the coordinate system.

We can see an $1 \times 1 \times 1$ *Bar-Joint Scaffolding* an its equation system without the shaded elements on Figure2, which describes the possible motion of a corresponding cubic framework.

The equation system is solvable, the set of the rigid body like motions is empty the consequence of the fixed joints on the ground and the lateral face.

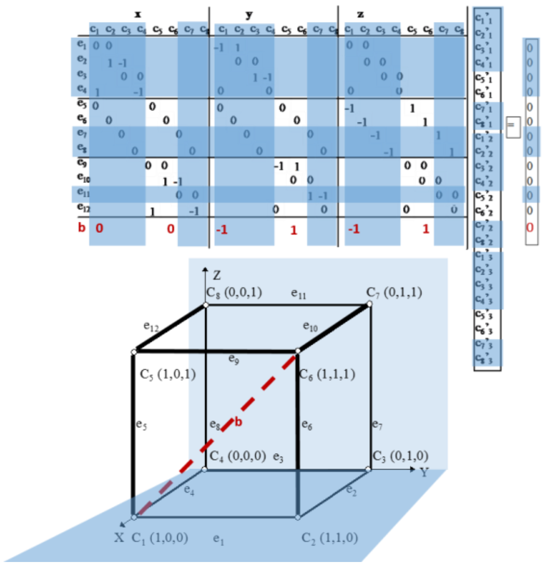

Figure 2

We can see The rigidity equation system of the $1 \times 1 \times 1$ Scaffolding framework. On the bottom, we can see a simple Scaffolding framework with one bracing item. We have to solve the above equation system without the shaded entries for determining the possible motions of the Scaffolding

$$
\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_{51}' \\ C_{61}' \\ C_{52}' \\ C_{62}' \\ C_{53}' \\ C_{63}' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

The row 1, 2, 4 and 5 imply the 0 displacements both of the joints into the direction X and Z the consequence of the shaded rows. Without the dashed diagonal brace, we get the C_{52}' is equal C_{62}', i.e. both joint has the same infinitesimal motion into the direction Y. However, the consequence of diagonal bar *b* there exists the row 6 in the equation system, i.e., $-C_{62}' = C_{63}' = 0$. Hence, all velocity is zero, i.e., our framework is rigid. Maxwell's rank condition is also satisfied. It is easy to understand that the constraint of the scaffolding implies the *0* displacements each of the joints of the Scaffolding into the direction *X* and *Z*, in the next paragraph 3.1 we present a geometrical justification for the general case.

If we decide the rigidity with Maxwell's rule, the complexity of the Scaffolding problem remained same, since the number of the variety is not decreased indeed. However, we know the number of the different displacement strongly decreased.

3 Results

The bracing problem of the bar-joint Scaffolding with short diagonal was considered in [39], in this paper, we use the name Bar-Joint Scaffolding instead of the name Annex Building for our framework because the name of Annex Building refers to the application of architecture only. These results for long diagonals is general than the earlier and is widely applicability. This result is useful from an algorithmic point of view. The results would be generalizing for higher dimensions similarly to [39].

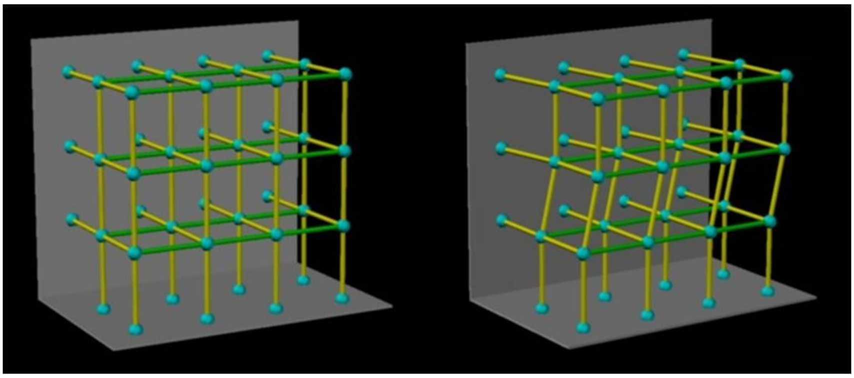

Figure 3

A motion of a scaffolding framework.

On the figure, we can see a 2×3×3 scaffolding framework without bracing elements. The intersection of the base face and the lateral face is baseline. In the beginning, the joints can move in the direction of the base line. Right, we can see a possible displacement some of the joints. Hence, this framework is not rigid.

3.1 Motion of the Bar-Joint Scaffolding

In the Scaffolding, the intersection of the base face and the lateral face is called baseline. It is the axis Y. Let a bar be parallel to the axis Y. If one of its endpoints have an infinitesimal motion in the Y direction, then the other endpoint of this bar has the same infinitesimal motion in the same Y direction, due to the rigidity of the bar. Initially, there is no X or Z motion component (infinitesimal displacement) of the joint, the consequence of the rigidity of the bar, which transmits these displacements to the lateral wall, and the ground. Let the joints of a not necessarily right prism are $A_1A_2A_3$ the bottom face and $B_1B_2B_3$ the top face, let the edges be bars with unity length, except A_1A_2 and B_1B_2. This prism framework can move in a way, that the bars A_iB_i, are not necessarily parallel, *where* i= 1, 2, 3. The critical assumption is that the two bars of Prism framework, A_1B_1 and A_2B_2 are parallel. The prism framework is not degenerate if the bar A_3B_3 is in the same half-space that is determined by $A_1B_1A_2B_2$ plane. The prism framework is not degenerate if the bar A_3B_3 is inside one of the half-space, which are determined by $A_1B_1A_2B_2$ plane, i.e. there is no common point between the section A_3B_3 and the plane $A_1B_1A_2B_2$.

The length of the bars A_1A_2, B_1B_2 is less than 2, and they are also parallel, but the exact length of them is not relevant in this case. We could leave the A_1A_2, B_1B_2 bars if we suppose that A_1B_1 and A_2B_2 are parallel.

Theorem 3.1: If the bar A_1B_1 and the bar A_2B_2 are parallel, and the plane $A_1B_1A_2B_2$ does not intersect segment A_3B_3 in the non-degenerated prism framework, then the bar A_3B_3 is also parallel to the bar A_1B_1 and the bar A_2B_2.

Proof:

A_3 is on the circle line C_A, which is the intersection of the sphere with center A_1 and radius A_1A_3 and the other sphere with center A_2 and radius A_2A_3. Similarly, B_3 is on the circle line C_B, which is the intersection of the sphere with center B_1 and radius B_1B_3 and the other sphere with center B_2 and radius B_2B_3. Circle C_A is translated to C_B by vector $\boldsymbol{A_1B_1}$. Hence, the translated joint A_3 by vector $\boldsymbol{A_1B_1}$ is a possible space for B_3. Indirect we assume there is another possible place for B_3, let that be B_{03}, it is the reflection of B_3 to the plane $B_1B_2A_3$. The point B_{03} is then below the $A_1B_1A_2B_2$. Hence, it is not suitable for B_3.■

If we suppose that the motions of the joints of the prism are continuous, and not so large, that the positions of the joints are not degenerate, then we can describe the motion of the framework with some variables.

A joint $J_{i;j;k}$ in the Scaffolding building has an infinitesimal motion in the direction of axis *Y*, then for any $1 \leq p \leq m$ every joint $J_{i;p;k}$ has the same infinitesimal motion component because they are connected with rigid bars which are parallel to the Y direction. The motion does not have a component in the X, or Z-direction since then the lateral wall or the ground would have the same infinitesimal motion.

Denote by $s_{i;k}$ the set of the bars that connect joint $J_{i;p\text{-}1;k}$ to joint $J_{i;p;k}$, where $1 \leq p \leq m$. Denote by $y_{i;k}$ the second coordinate (*y)* of $J_{i;0;k}$. Naturally, $y_{0;k}$,= 0 and $y_{i;0} = 0$, because all of these joints are fixed either to the lateral wall or the ground.

Denote by $b(J_{i1;j1;k1}\ ;\ J_{i2;j2;k2}\ )$ the long diagonal brace bar between joint $J_{i1;j1;k1}$ and joint $J_{i2;j2;k2}$, where $j_1 \neq j_2$. We say long diagonal but is possible that some of the brace or all of them are no longer than the diagonal of a unit square.

Lemma: If there is a long diagonal brace $b(J_{i1;j1;k1}\ ;\ J_{i2;j2;k2}\ )$ then $y_{i1;k1} = y_{i2;k2}$.

Proof: About the rigidity of the large brace its endpoints have the same motion component into the direction of the axis of the bar, this implies the $y_{i1;k1} = y_{i2;k2}$, since the other motion components of the endpoints are zero. ■

In this case, the brace is useful for the infinitesimal rigidity of the framework, since if one of its endpoints has infinitesimal motion in the *Y*-direction imply that the other endpoint has the same infinitesimal motion as well, it is important that $j_1 \neq j_2$.

Corollary 3.1: If the second coordinates of two joints is equal, i.e., $j_1 = j_2$, then the long diagonal brace is perpendicular to the *Y* axis. In this case, the brace is useless for the infinitesimal rigidity of the framework, since infinitesimal motion in the *Y*-direction of one of its endpoints not imply, that the other endpoint has the same infinitesimal motion in the *Y*-direction.

3.2 Bracing Graph

Let us define the bracing graph.

Definition 2.2: $B(P_{i;k}; E)$ bracing graph of the Scaffolding building: Let $l_{i;k}$ correspond to points $P_{i;k}$ in the bracing graph, if $i \neq 0$ and $k \neq 0$. If i or k equal to 0 then $l_{i;k}$ corresponds to the same point $P_{0;0}$. If there is a $b(J_{i1;j1;k1} ; J_{i2;j2;k2})$ diagonal brace, between joints $J_{i1;j1;k1}$, $J_{i2;j2;k2}$, then $(P_{i1;k1}, P_{i2;k2}) \in E$, i. e. there is an edge between vertices$P_{i1;k1}$ $P_{i2;k2}$.

If will be more than one bracing bar in the case different j_1 or j_2; then we connect the corresponding points with multiple edges.

3.3 A necessary and sufficient condition

We give a necessary and sufficient condition for the infinitesimal rigidity of a Scaffolding if we use diagonal braces.

Theorem 3.2: The $l \times m \times n$ Scaffolding with some short or long diagonal braces is infinitesimally rigid, i.e. kinematically determinate, if and only if its bracing graph is connected.

Proof: The rigidity implies the connectivity of the bracing graph. The proof is indirect: Assume that the bracing graph is not connected, then there exists a component of the bracing graph, which does not contain $P_{0;0}$. The bars and joints of the corresponding $l_{i;k}$ have a (possibly infinitesimal) motion in the Y-direction. Hence, the assumption was false.

The proof of the other direction: if the bracing graph is connected, then we can get from $P_{0;0}$ to every $P_{i;k}$. Hence, the consequence of the Lemma every $y_{i;k} = y_{0;0} = 0$ for any i, k. Therefore, for any joint in the Scaffolding, there is no infinitesimal motion in the *Y-direction, there is no any infinitesimal motion in the X-direction,* and in the *Z-direction* that is the consequence of the fix lateral face and the fix planar ground respectively. ■

In Figure 4 we can see on the left-hand side the bracing graph of the right-hand side framework. The graph is not connected; it has two components. Hence, the upper part of the framework can move. If we want to make it rigid, we can use one further diagonal brace, which is corresponding to an edge connecting the two components of the bracing graph for example between bar b $(J_{1;2;1} ; J_{1;3;2})$ would be convenient.

Corollary 3.2: The number of the braces is independent of m. We need not less than $l \cdot n$ diagonal braces for the infinitesimal rigidity.

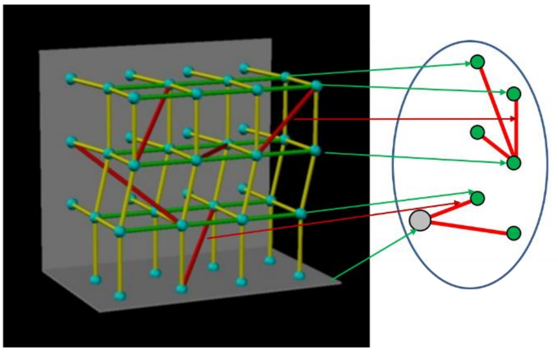

Figure 4

The bracing graph of the Scaffolding

We can see 2×3×3 scaffolding framework with five bracing elements. The arrows represent the construction of the bracing graph. On the right-hand side of we can see the bracing graph of the framework. This graph is not connected. Hence, some elements of the structure can move, but the first floor is rigid.

3.5 Non-periodic Scaffolding

The reviewer asks about non-periodic structures. In this paper, it is critical, that at the beginning of the motion, all of the joint can move only to the same (Y) direction. This assumption is satisfied in that case when the length of certain bars are different, and certain bars are equal. Hence, we could change the heights of the bars of any floors in the Z direction. Similarly, we can modify the length of the bars the other two directions.

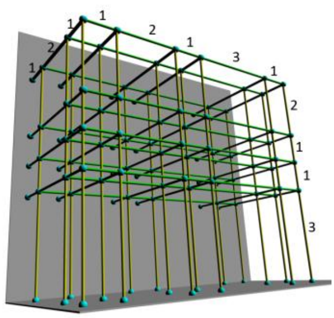

Figure 7

Non-periodic scaffolding

On the picture, we can see a Non-periodic Scaffolding in the X direction with variable bar lengths 1, 2, 1. In the Y direction 1, 2, 1, 3, 1 are the lengths of the bars respectively. In the Z direction with length 3, 1, 1, 2, these are also the heights of the floors.

Hence, we could characterize the rigidity of the framework that its length of bars are signed on Figure 7, naturally the length of the vertical bars on some of the floor are equal, but not necessarily integer.

4. Applications

4.1 Building from Scaffolding

As an example of a scaffolding can see on, Fig. 8. was considered by FEM analyses in (Chandrangsu, K.J.R. Rasmusse) and the loads and displacements characteristic form the nonlinear analyses was consistent with their experiments, which were controlled by increasing the applied load from direction *Z*.

According to our consideration, this framework is not scaffolding, since its joints not connected to a fix lateral wall. We denote these structures as 3 story building, shortly building. The rigidity of braced bar-joint building was considered in Radics, Recski with the tools of combinatorial optimization also.

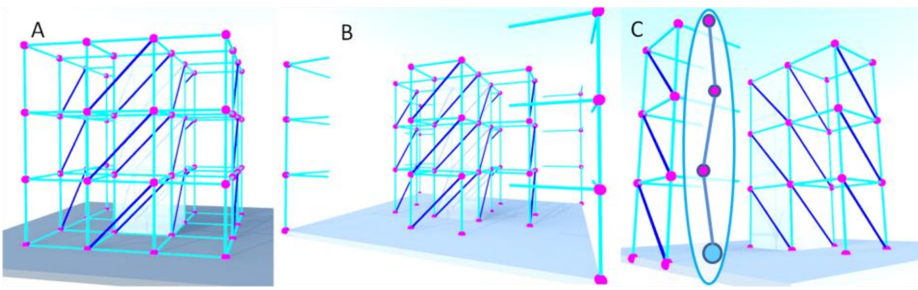

Figure 8

The building is built from scaffolding. On the picture, we can see a braced 3×3×3 building, it has one Core in the middle, that holds four Expanded Cores, which hold the remained parts of the building. In the Picture C we can see the bracing graph of the Expanded Cores.

For the stiffness of high building the designer generally use core, or cores, which made from reinforced concrete for its stability. On the considered building the inner 1×1×3 square based transparent prism may regard as the core of this building (A), since its sides are full braced, therefore it is rigid. We translate the joints of the corners with their connected columns and bends (B). The remained parts, denote them expanded core of the building, there are four on the middle parts of picture B, are also 1×1×3 square-based prisms may regard as scaffolds since they stand on the ground and connect to the core. We can see on picture C the Core with an Expanded Core in front and beside the elements of Scaffolding which create the next Expanded Core. Their rigidity graph is connected, the consequence of our *Theorem 3.2*, therefore they are rigid.

The remained parts (the translated corners) of the building connect with lateral walls to the Expanded Cores of the building which are perpendicular to each other, therefore the all braced bar-joint building is rigid. The stiffness of the expanded core is the consequence of the braced scaffold rigidity.

For the sake of accuracy, our redefined building are denoted the bar-joint prefix since it is different from the general engineering practice.

Definition 4.1. The $l \times m \times n$ Bar-Joint Building is a $l \times m \times n$ cube grid framework and its joints on the base $l \times m$ face are attached to fixed joints in the planar ground, which is the *X-Y* plane of the coordinate system.

In this case the joints on the back lateral $m \times n$ face are not connected to fixed joints to the lateral wall that was the *Y-Z* plane of the coordinate system in case of Bar Joint scaffolding.

Definition 4.2. The Bar-Joint Core is an $a \times b \times n$ Bar-Joint Building as a rigid part of an $l \times m \times n$ Bar-Joint Building, where $a \le l$, and $b \le m$.

If the Building is fully braced, i.e. its all diagonals is a brace than it will be rigid, naturally it is not necessary for the rigidity of the building.

Definition 4.3. The Expanded Cores of a Bar-Joint Core of a Bar-Joint Building is that parts of that may regard as a Bar-Joint Scaffolding and there is mutual lateral wall with the Bar-Joint Core as the rigid part of the Bar-Joint Building,

Using the former results we provide sufficient assumption for the rigidity of the braced Bar-Joint Building.

Theorem 4.1: If the braced Expanded Cores of the Bar-Joint Building with a Bar-Joint Core are rigid, then the Bar-Joint Building is rigid.

Proof: the rigidity of the $k \times l \times m$ Bar-Joint Building is the consequence of the previous sentences which proved the rigidity of the $3 \times 3 \times 3$ Bar-Joint Building on Picture A in Figure 8.■

Corollary 4.1: This theorem allowed to use long diagonals for the rigidity of the expanded scaffold, but the shorts are better for economy point of view.

Corollary 4.2: The number of the necessary braces for the rigidity of the $l \times m \times n$ Bar-Joint building with $1 \times 1 \times n$ Bar-Joint Core is $n(l+m-2)$ independently from the place of the core.

The reverse theorem is not true. If one of the expanded cores is not rigid, the building can still be rigid. We regard the 3-th floor of the 3×3×3 Bar-Joint Building (Fig. 9A) and place its brace of the Expanded Cores from one of the lateral wall to the horizontal ceiling (Fig. 9B), it is remained rigid as we can see the bracing graph is changed, but remained connected, hence the Expanded Cores are rigid and also the Building. If we translate the brace one of the ceiling to right, then the structure remained rigid but the braces of the front side expanded core is lost, therefore this expanded core as scaffolding is not rigid.

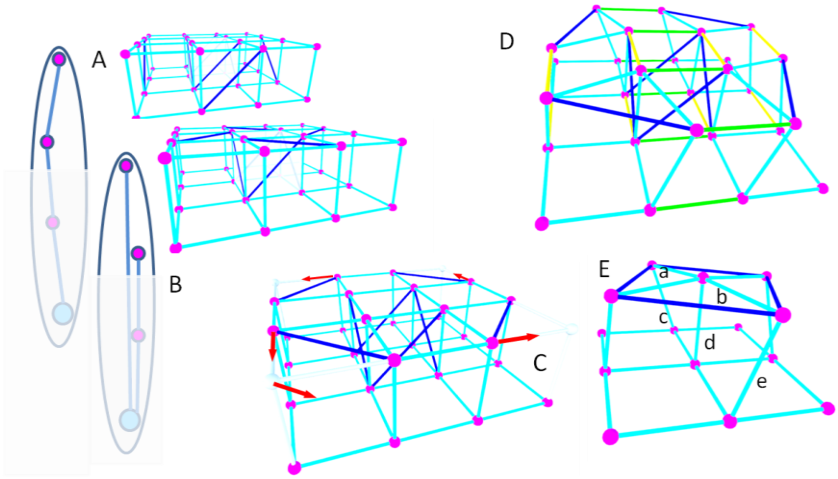

Figure 9

On the picture A, we can see the 3-rd floor of the braced 3×3×3 building. it has one Core in the middle, that holds four Expanded Cores, which hold the remained parts of the building. In the Picture C we can see the bracing graph of the Expanded Cores.

Corollary 4.3: The rigid slabs elements are allowed to use as braces horizontal squares for the rigidity of the Expanded Core.

In this case the slabs have to remain in its position, i.e. they not allowed to translate from the Expanded Core. If we translate the braces we may lost the stability as we can see on Figure 9C. In this case all of the horizontal bracing elements are in the corner of the ceiling, and there are some joint which have infinitesimal motion to the direction of the arrows, since the joints in the upper corner and their bars are irrelevant as constraints, therefore we ignored it.

There is no finite motion in the former case. Contrary to the statement, we suppose there is a finite movement as we can see on Figure 9D. Since the green colored bars are equal and parallel during this motion see *Theorem 3.1*, the left part of the structure can translate next to the right part of the structure erasing the green colored bars. Similarly we translate the back part of the structure along the yellow colored bars to the front part of the structure we get the Structure on the picture E in Figure 9. The upper part of this structure with the braces consists of four Isosceles Right Triangles, which would only be able to move if one of the pairs of opposing bars for example *a* and *b* remain parallel, in this case the connected vertical bars *c*, *e* are fixed since *d* is fixed originally, which imply the other two vertical bars are fixed also. The rotational stiffness of the building that braced according to our theorem is the consequence of the translational stiffness of all joints to both of the direction X and Y.

4.2 The Scaffolding Rigidity in the Standards

The consequence of our result could be further considered of the OSHA 3150 2002 (Revised) [68]. The scaffold one of its pictures is not rigid in Figure 9.

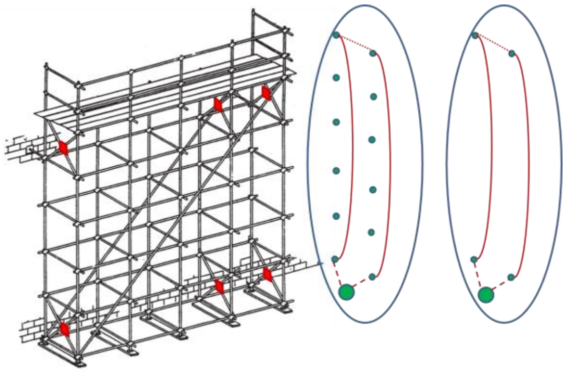

Figure 9. Infinitesimally not rigid Scaffolding. OSHA 3150 2002 scaffold framework is on the left [71]. Disregarding the crash barrier of the scaffolding we may use two bracing graphs; the first one is not connected, the second graph is connected the consequence of the Scaffold Planks.

If the applied joint connections are regarded to ball joint connection, then framework is a 1×4×5 scaffolding, we can see its rigidity graph of the left-hand side framework in the middle of Figure 9. If the links of the vertical bars (Posts) of the framework are regarded to rigid connection, then the framework may be regarded as a 1×4×1 scaffolding, and the rigidity graph of this framework is on the right-hand side of the picture. The large node corresponds to the ground and the lateral wall, the dashed lines connections is the consequence of the sticking friction between the ground and the sleeper of the scaffolding. Reassuringly, none of them is connected. Hence, the consequence of Theorem 2.2, they are infinitesimally not rigid even if we connect all joints which are near to the lateral wall to this wall with short bars and ball joint. In the latter case, the diagonal Cross Bracing in the signed squares is unnecessary the consequence of Corollary 2.3.2. Without these short bars, the structure cannot be considered as a Scaffold. In this case, the looseness of the connections of the vertical elements, and the lost connection to the lateral wall there are infinitesimal horizontal motions not just to the lateral (*Y*), but to the *X* direction also which imply finite motions.

Notwithstanding, in the second case, the framework may regard rigid if the next assumption are satisfied:

The Scaffold Planks are tight enough that they regarded as diagonal brace, signed by the dotted line between the top of the two nodes;

The joints of the inner side vertical elements are fixed at least in the direction *X*.

Since these conditions are missing from the OSHA Guide, the revision of this standard is to be wished, similarly to (Prabhakaran) [50].

5 The deciding of the rigidity of the Scaffolding

5.1 The computation of the rigidity of Scaffolding

Our model describes the structures that are periodic as in [39] using arbitrarily long diagonals. These make the system more usable, making them promising candidates for civil engineering structures, material science, nanotechnology and biomedical applications. The possible algorithm consists of the next steps:
Take a Scaffolding type framework with some bracing elements

1. Choose an adequate data structure for joints, bars and bracing elements
2. Decide the framework is a Scaffolding
3. Decide the bracing graph nodes and edges
4. Decide the connectedness of the bracing graph
5. Describe the possible infinitesimal motions of the joints of the structure

With this procedure, the rigidity properties of variety of structures based on Scaffolding type framework could be identified.

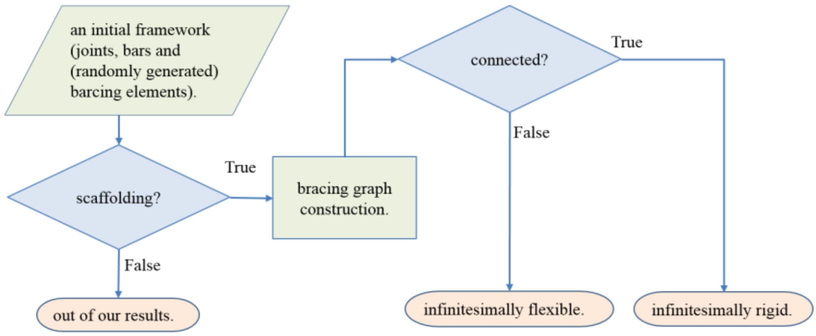

Figure 10. Flowchart of the rigidity of Scaffold algorithm. The construction of the bracing graph is different in the of the recess framework.

The Fig. 10 shows a simplified flowchart of this algorithm. Let N be the number of the nodes plus the number of the edges of the bracing graph. The time complexity of the rigidity algorithm of the above framework braced with diagonals is linear with the size of the graph, because the time complexity of checking connectivity is of the order of N.

Using Theorem 2.2, we can decide the rigidity of a Scaffolding of the order of $(ln)^2$ steps, because an upper bound for the number of steps of the test, to decide if a graph is connected, is the number of the possible edges. For instance, the number of the joints are lmn in the case of the Scaffolding, and the size of the auxiliary graph of the framework is ln, and hence the time complexity of the proposed algorithm, is $(ln)^2$. While according to Maxwell, the number of operation would be of the order of $(lmn)^3$ using Gaussian elimination. Using Maxwell method for the upper bound of the counting size of a cubic size Scaffold is approximately 50 for daytime. In the case of our result, the upper bound of the Scaffold size is near 10000 regarding similar terms.

5.2 The computation of the rigidity of Building

5.3 Computing the safety of Scaffolding

Real frameworks networks consist of redundant connections. The redundancies are important the safety of the rigidity of the network. The redundancies of the structural elements in consequence of its uncertainty may be saved the building and its serviceability under catastrophes or hazardous events.

Definition 5.1: The framework is safety if some of the elements collapse while the remainders have kept the rigid structure yet.

The bracing graph is connected in the case of rigid structures. We have to demand the redundancies of the bracing elements. Hence we increase the connectivity of the bracing graph. Similar problems are significant and well-studied optimization problems in graph theory, and network analysis. The connectivity augmentation problem is the next in our case: given a bracing graph and a positive integer k, we find a minimum number of new edges in the bracing graph, that the results will be k edge-connected. Hence, if we remove arbitrary k−1 edges that corresponded to the removed bracing element the remained object not collapsed. The connectivity augmentation helps us to increase the safety of an already existing network by adding an optimal number of new connections for the rigidity of Scaffolding.

It is an open question even if the graph G to be augmented is (k-1) vertex-connected. Polynomial algorithms have been developed only some cases [72]. In network design, it is often of interest to know how sensitive a particular property of a network is to changes in the graph structure, like the removal or failure of edges. We focus on the edge-connectivity of a graph. The connectivity interdiction asks to decrease the edge-connectivity of a graph maximally by removing a limited set of edges. We can ask how many braces could be taken away at most before the graph disconnects. Hence, our framework would collapse. If the edge connectivity is k then we could take away k brace, the graph will become disconnected (not all of k brace are good for the disconnectivity), i.e. the framework will be a mechanism.

5.3 Computing the safety of Building

6. Discussions

6.1 The boundary of the rigidity optimization of Scaffolding

Our graph theoretical model determines the flexibility or rigidity of the ideal structure of Scaffolding, with undeformed bars and freely rotatable joints. Maxwell's result counts the degree of freedom of the framework, and it solves the problem without the constraint that came from the lateral wall and the ground as we have seen in the 2.2.1. It is expected that overall stiffness of a biological network consisting of rigid and soft elements be determined mainly by the elastic elements, which are the bracing elements [54-59, 62-66]. In this case, the network stiffness would be insensitive to slight changes in the rigidity of the stiff items that would be for example the filaments [55, 58, 62]. Structurally related hierarchical tissue systems exist throughout the human body, for instance in partially mineralized tissues at tendon-to-bone attachments consisting of collagen cross-linked by solid mineral particles. The infinitesimal rigidity of fibrils in bundle structures is similar as scaffolding as we can see in section 4.1 and 4.2. The latter application could be a good model for the composite structures [60-62].

Our results provide strategies that maximize the number of the element of fragments and minimize preservation of intact cellulosic fibrils that structure are also similar to the scaffolding if the assumption is satisfied for example in the case of arboreal plants [56-58]. We can find the optimal structures that useful for rigidity point of view or contrary could move (developing). This result is helpful to the input of simulation of the real elastic behavior of the elements in a given structure, for example; we can test the rigidity of the structure before the considerations of the Finite Eleeemmments Methods. Hence, this model could be a good candidate for the future work in cell biology. Using the results of flexibility and rigidity of Scaffolding, we can predict the motion of a structure of a ten thousand individual bars in the structures, thanks to the Graph-theoretic and graphic reduction of the complex system.

7 Conclusions

In this paper, we consider frameworks that consist of deterministic (not stochastically chosen), fixed length bars that could rotate independently around the connected common joint. Our frameworks are grid-like finite structures. The results of this paper:

- A new characterization of scaffolding with arbitrary lengthed bracing elements. The characterization is very expressive that we can see in Figure 2, Figure 3, and Figure 5, the Scaffolding using some braces is rigid if the projection of joints and bracing elements (without any bars) to Y-direction, i.e. the direction of the projection is parallel to the baseline, is connected.
    - The main contribution of our result is the Theorem 2.2 that provide an efficient algorithm deciding the rigidity of Scaffolding, frameworks.
    - The connectivity of the bracing graph of our frameworks characterizes the flexibility and the rigidity of the structure.
- We could estimate using this method how many bracing elements sufficient for the rigidity i.e. for the kinematically determinate systems. Our results provide strategies that evaluate the number of the bracing elements of fragments and minimize preservation of intact parallel bars.
- The rigidity analysis of the structure that consists of ideal bars and joints elements provide very fast results to the input of the simulation of the real elastic behavior of the elements in a given structure, for example; we can test the rigidity of the structure before the considerations of the Finite Elements Methods.
- We suggest the revision of the sample scaffolding of the OSHA standard.

- The kinematical characterization taken provides a powerful model for exploring the infinitesimal mobility of these structures and gives a new approach to understanding the nanoscale behavior of biological materials as fibrils and other fibrous polymers.

Acknowledgments.

Avinash M. Nafday Consequence-based structural design approach for black swan events Structural Safety, Volume 33, Issue 1, January 2011, Pages 108-114

R.G. Beale Scaffold research — a review Journal of Constructional Steel Research, 98 (2014), pp. 188-200

H. Zhang, T. Chandrangsu, K.J.R. Rasmussen Probabilistic study of the strength of steel scaffold system Structural Safety, 32 (2010), pp. 393-401

Gian Paolo Cimellaro Marco Domaneschi Stability analysis of different types of steel scaffolds Engineering Structures Volume 152, 1 December 2017, Pages 535-548

Kaveh A, BolandGerami A (2017) Optimal design of large-scale space steel frames using cascade enhanced colliding body optimization. Struct Multidiscip Optim 55(1):237–256

Kaveh A, Ghafari MH, Gholipour Y (2017) Optimal seismic design of 3D steel moment frames: different ductility types. Struct Multidiscip Optim:1–16

Kaveh, A. & BolandGerami, A. Struct Multidisc Optim (2017) 55: 237. https://doi.org/10.1007/s00158-016-1494-2

Ahmadi, B., Jalalpour, M., Asadpoure, A. et al. Struct Multidisc Optim (2018) 58: 2533. https://doi.org/10.1007/s00158-018-2035-y

Lu, H., Gilbert, M. & Tyas, A. Struct Multidisc Optim (2018) 58: 677. https://doi.org/10.1007/s00158-018-1921-7

T. Chandrangsu, K.J.R. Rasmusse Structural modeling of support scaffold system

Journal of Constructional Steel Research, 67 (2011), pp. 866-875

Mu Zhu, Yang Yang, James K. Guest, Michael D. Shields, Topology optimization for linear stationary stochastic dynamics: Applications to frame structures Structural Safety, Volume 67, July 2017, Pages 116-131

Ties to façade

Bracing is commonly used to provide resistance to lateral forces in building structures. However, traditional bracing design approaches appear not to be underpinned by clear fundamental principles. Here, theoretically optimal arrangements of bracing members are sought for pre-existing building frames, already designed to carry gravity loads. For sake of simplicity existing frame elements are assumed to be capable of carrying additional loads and three types of bracing are considered: tension only bracing, bracing intersecting only at the corners of the existing frame, and unconstrained optimal bracing, where bracing elements can intersect at any location.

References

1. Stauffer D. Introduction to Percolation Theory. Taylor & Francis, London/Philadelphia, 1985.
2. Moukarzel C, Duxbury PM, Leath PL. Infinite-cluster geometry in central-force networks. Phys. Rev. Lett. 1997;78,8:1480–1483
3. Zhang H, Chandrangsu T, Rasmussen KJR, Probabilistic study of the strength of steel scaffold systems, Structural Safety 2010;32,6:393-401.
4. Zhang L, Rocklin DZ, Chen BG, Mao X. Rigidity percolation by next-nearest-neighbor bonds on generic and regular isostatic lattices. Phys Rev E 2015;91:032124.
5. Ellenbroek WG, Hagh VF, Kumar A, Thorpe MF, van Hecke M. Rigidity Loss in Disordered Systems: Three Scenarios. Phys Rev Lett 2015;114:135501.
6. Dove MT. Introduction to lattice dynamics. Cambridge topics in Mineral Physics and Chemistry, C.U.P.; 1993.
7. Zalka KA. Global Structural Analysis of Buildings. E & FN Spon, London, 2000
8. Chilton J. Space Grid Structures. Architectural Press, Butterworth-Heinemann, London, 2000.
9. Yu WK, Chung KF, Chan SL. Structural instability of multi-storey door-type modular steel scaffolds, Engineering Structures 2004;26,7:867-881.
10. Kaliszky S, Lógó J. Optimal design of elasto-plastic structures subjected to normal and extreme loads. Computers & Structures 2006;84:1770–9.
11. He Y, Zhou X, Liu J. Dynamic property and seismic performance of square-based membranous latticed shells, Soil Dynamics and Earthquake Engineering 2012;43:25-32.
12. Bacigalupo A, Gambarotta A. Multi-Scale Strain Localization Analysis of a Layered Strip with Debonding Interfaces. International Journal of Solids and Structures 2013;50:2061-2077.
13. Mohr DP, Stein I, Matzies T, Knapek CA. Redundant robust topology optimization of truss. Optim Eng 2014;15:945–72.
14. Yu X, Ji T, Zheng T. Relationships between internal forces, bracing patterns and lateral stiffnesses of a simple frame. Engineering Structures 2015; 89:147-161.
15. Patil DM, Sangle KK. Seismic Behaviour of Different Bracing Systems in High Rise 2-D Steel Buildings, Structures 2015;3:282-305.

16. Fateh A, Hejazi F, Jaafar MS, Karim IA, Adnan AB. Design of a variable stiffness bracing system: Mathematical modeling, fabrication, and dynamic analysis. Soil Dynamics and Earthquake Engineering 2016;80:87-101.
17. Liu H, Jia L, Wen S, Liu Q, Wang G, Chen Z. Experimental and theoretical studies on the stability of steel tube–coupler scaffolds with different connection joints, Engineering Structures 2016;106:80-95.
18. Maxwell J.C. On the calculation of the equilibrium and stiffness of frames. Philosophical Magazine. 1864;27:294–299
19. Pellegrino ZS. Foldable bar structures. Int. J. Solids Struct. 1997;34:1825–1847.
20. Recski A. Matroid Theory and its Applications in Electric Network Theory and in Statics. Akadémiai Kiadó, Budapest and Springer, Berlin. 1989.
21. Graver J, Servatius B, Servatius H. Combinatorial rigidity, Graduate Texts in Mathematics, vol. 2. Amer. Math. Soc.; 2008.
22. Guest SD, Hutchinson JW. On the determinacy of repetitive structures. Journal of the Mechanics and Physics of Solids. 2003;51:383–391.
23. Bolker ED, Crapo H. How to brace a one-story building. Environ. Plann. B.1977;4:125-152.
24. Bolker ED. Bracing rectangular frameworks II. SIAM J. Appl. Math. 1979;36,3:491-508.
25. Bolker ED, Crapo H. Bracing rectangular frameworks I. SIAM J. Appl. Math. 1979;36,3:473-490.
26. Kaveh A. Graph theoretical methods for efficient flexibility analysis of planar trusses. Computers & Structures 1986;23,4:559-564.
27. Kaveh A. Matroids in structural mechanics. Computers & Structures 1993;47,1:169-174.
28. Kaveh A. Topological transformations applied to structural mechanics. Computers & Structures. 1997;63,4:709-718.
29. Gaspar Zs, Radics N, Recski A. Square Grids with Long Diagonals, Optimization Methods and Software. 1998;10:217-231.
30. Gaspar Zs, Radics N, Recski A. Rigidity of square grids with holes, Computer Assisted Mechanics and Engineering Sciences. 1999;6:329-335.
31. Recski A. Applications of combinatorics to statics—a second survey. Discrete Math 1992;108:183–8.
32. Radics N, Recski A. Applications of combinatorics to statics rigidity of grids, Discrete Applied Math. 2002;123:473-485.
33. Nagy G Tessellation-like Rod-joint frameworks, Annales Univ. Sci. Budapest. 2006;49:3-14.
34. Mitschke H, Robins V, Mecke K, Schroder-Turk GE. Finite auxetic deformations of plane tessellations. Proc R Soc A Math Phys Eng Sci 2013;469:20120465–20120465.

35. Ito Y, Kobayashi Y, Higashikawa Y, Katoh N, Poon S-H, Saumell M. Optimally bracing grid frameworks with holes. Theor Comput Sci 2015;607:337–50.
36. Nagy Kem G. Bracing rhombic structure by one-dimensional tensegrities. Meccanica 2017;52,6,1283–1293.
37. Recski A. Bracing cubic grids—a necessary condition. Discrete Math 1988;73:199–206.
38. Nagy Gy. Diagonal bracing of special cube grids. Acta Technica Academiae Scientiarum Hungaricae. 1994;106,3-4:256-273.
39. Nagy Gy. Rigidity of an annex building. Structural and Multidisciplinary Optimization. 2001;22,1:83-86.
40. Radics N. Rigidity of multi-story buildings. DiscretAppl Math 2001;115:187–98.
41. Obiała R, Iványi P, Topping BHV. – Genetic algorithms applied to partitioning for parallel analyses using geometric entities. Comput. Fluid Solid Mech. 2003;2003:2078–81.
42. Guest SD, Hutchinson JW. On the determinacy of repetitive structures. Journal of the Mechanics and Physics of Solids. 2003;51:383–391.
43. Hutchinson RG, Fleck NA. The structural performance of the periodic truss, Journal of the Mechanics and Physics of Solids. 2006;54:756–782.
44. Stephen NG. Transfer matrix analysis of the elastostatics of one-dimensional repetitive structures. Proc R Soc A Math Phys Eng Sci 2006;462:2245–70. doi:10.1098/rspa.2006.1669.
45. Kaveh A. Topological properties of skeletal structures. Computers & Structures 1988;29:403–11.
46. Kaveh A, Farhoudi N. Dolphin monitoring for enhancing metaheuristic algorithms: Layout optimization of braced frames. Computers & Structures 2016; 165:1-9. ISSN 0045-7949,
47. Lógó J. On the optimal layout of structures subjected to probabilistic or multiply loading. Struct MultidiscipOptim 2013;48:1207–12.
48. Owen JC, Power SC, Frameworks, symmetry and rigidity. Inter. J. Computational Geometry and Applications 2010;20:723–750.
49. Malestein J, Theran L. Generic combinatorial rigidity of periodic frameworks. Advances in Mathematics. 2013;233,1:291–331.
50. U. Prabhakaran, R.G. Beale, M.H.R. Godley, Analysis of scaffolds with connections containing looseness, Computers & Structures, Volume 89, Issues 21–22, November 2011, Pages 1944-1955,
51. Jordán T, Domokos G, Tóth K. Geometric Sensitivity of Rigid Graphs. SIAM Journal on Discrete Mathematics. 2013;27,4:1710-1726.
52. Dóbé P, Domokos G. Computer-Aided Geometric Sensitivity Analysis of Plane Trusses with a Square Grid Topology. Period Polytech Electr Eng Comput Sci 2014;58:55–68.

53. Błazik-Borowa E, Gontarz J. The influence of the dimension and configuration of geometric imperfections on the static strength of a typical façade scaffolding, Archives of Civil and Mechanical Engineering 2016;16,3:269-281,
54. Nagy Kem G. Flexibility and rigidity of cross-linked Straight Fibrils under axial motion constraints. J Mech Behav Biomed Mater 2016;62:504–14.
55. Ingber DE. Cellular tensegrity: defining new rules of biological design that govern the cytoskeleton. J Cell Sci 1993;104.
56. Fratzl P, Weinkamer R. Nature's hierarchical materials. Prog Mater Sci 2007;52:1263–334.
57. Saitoh K, Ohno H, Matsuo S. Structure and Mechanical Behavior of Cellulose Nanofiber and Micro-Fibrils by Molecular Dynamics Simulation. Soft Nanosci Lett. 2013;3:58–67.
58. Ilnicka A, Lukaszewicz JP. Discussion Remarks on the Role of Wood and Chitin Constituents during Carbonization. Front Mater 2015;2:1–6.
59. Sjölund J, Karakoc A, Freund J. Effect of cell geometry and material properties on wood rigidity. Int J Solids Struct 2015;62:207–16.
60. Väkiparta M, Yli-Urpo A, Vallittu PK. Flexural properties of glass fiber reinforced composite with multiphase biopolymer matrix. J Mater Sci Mater Med 2004;15:7–11.
61. Säilynoja ES, Lassila L, Garoushi S, Vallittu PK. The effect of fibers length on the fracture toughness of short fiber reinforced composites. Dent Mater 2013;29.
62. Gautieri A, Vesentini S, Redaelli A, Buehler MJ. Hierarchical structure and nanomechanics of collagen microfibrils from the atomistic scale up. Nano Lett 2011;11:757–66.
63. Park YB, Cosgrove DJ. A revised architecture of primary cell walls based on biomechanical changes induced by substrate-specific endoglucanases. Plant Physiol. 2012;158:1933-1943.
64. Solar M, Buehler MJ. Deformation behavior and mechanical properties of amyloid protein nanowires. J Mech Behav Biomed Mater 2013;19:43–9.
65. Ahmadzadeh H, Smith DH, Shenoy VB. Viscoelasticity of Tau Proteins Leads to Strain Rate-Dependent Breaking of Microtubules during Axonal Stretch Injury: Predictions from a Mathematical Model. Biophysical Journal 2014;106,5:1123-1133,
66. Zhang Z, Liu B, Zhang YW, Hwang KC, Gao H. Ultra-strong collagen-mimic carbon nanotube bundles. Carbon N Y 2014;77:1040–53.
67. Depalle B, Qin Z, Shefelbine SJ, Buehler MJ. Influence of cross-link structure, density and mechanical properties in the mesoscale deformation mechanisms of collagen fibrils. J Mech Behav Biomed Mater 2015;52:1–13.
68. Erdős P, Rényi A. On random graphs I. Publ. Math. Debrecen 1959;6:290-297.

69. Erdős P, Rényi A. On the evolution of random graphs. Publ. Math. Inst. Hungar. Acad. Sci. 1960;5:17-61.
70. Wikipedia. Components of the eukaryotic cytoskeleton, Retrieved June 1, 2016, from https://en.wikipedia.org/wiki/Microtubule#/media/File:FluorescentCells.jpg
71. OSHA U.S. Department of Labor: A Guide to Scaffold Use in the Construction Industry. Small Business Safety Management Series OSHA 3150 2002 (Revised). Retrieved 10. 05, 2016, from https://www.osha.gov/Publications/osha3150.pdf
72. T. Igusa, S. G. Buonopane, B. R. Ellingwood. Bayesian analysis of uncertainty for structural engineering applications Structural Safety, 24, 2–4, 165-186
73.
74.
75. Frank A. Connections in Combinatorial Optimization, Oxford University Press, Oxford Lecture Series in Mathematics and its Applications, 38, 2011.
76.