Brief note

Struct Multidisc Optim 22, 83–86

Rigidity of an annex building

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Abstract An annex building is a $k \times \ell \times m$ rectangular cube grid rod-joint framework and its joints on the base face and on the lateral face are attached by fixed joints to the planar ground and to a planar lateral wall, respectively. Inserting some diagonal braces the framework will be rigid. We give a fast algorithm for the annex building bracing problem.

Key words grids, space, rigidity, complexity, graphs

1 Introduction

One of the simplest structures in statics is the framework. A framework consists of rigid rods connected by rotatable joints.

Definition 1.1. A framework is rigid if any continuous motion of the joints that keeps the length of every rod fixed, also keeps fixed the distance between every pair of vertices in the framework.

In statics rigidity does not even allow infinitesimal motions. A framework is infinitesimally rigid if it is rigid and does not have any infinitesimal motion. A framework in the plane and a joint that has infinitesimal motion in the direction of the arrows are shown in Fig. 1.

Consider a rod-joint framework in the form of a $k \times \ell$ rectangular grid. This framework is not rigid. The question is how one can add diagonal braces to some of the squares of the grid framework in such a way as to make the framework rigid. This square grid bracing problem was solved generally by Maxwell (1864) but in his result

Received January 21, 2000

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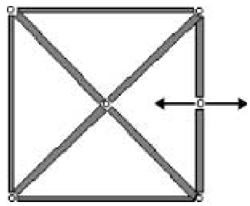

Fig. 1 Infinitesimal motion

the time complexity of deciding the rigidity is $O[(k\ell)^{2.4}]$. The result of Bolker and Crapo (1979) is better from an algorithmic point of view, because the size of the auxiliary graph of the framework, and hence the time complexity of the proposed algorithm, is $O(m+n)$.

The corresponding problem for cubic grids is open. There are results from Bolker (1977, 1979) but these are not satisfactory from the algorithmic point of view and Recski (1988/1999) gave a necessary condition. This author described the bracing of a special cube grid problem (Nagy 1996). The word "special" means that we assume that the opposite rods of square faces of any cube are parallel during any motion of the vertices.

A one-storey building is a $k \times \ell \times 1$ cube grid framework whose joints of the bottom $k \times \ell$ face are attached to fixed joints to the ground. Bolker and Crapo (1979) also have a result for the problem of the one-storey building. This result was generalized to the case of t-storey buildings by Radics (1999).

2 Annex building bracing

2.1 Annex building

Consider a $k \times \ell \times m$ rectangular cube grid. There are ball joints in the grid points and there are rigid rods in the grid edges. Thus we have a cube grid framework. The length of the rods is unity. This framework is not rigid. The question is how one can add diagonal braces to some of the unit squares of the grid in such a way as to make the framework rigid. This problem was solved generally by Maxwell (1864) but using his result the time complexity of deciding the rigidity is $O(n^{2.4})$, where n is the number of the joints. We are interested in an algorithm with time complexity $O(n)$ or less. Such an algorithm is not known for the cube grid framework.

The annex building is a $k \times \ell \times m$ cube grid framework and its joints on the base face are attached to fixed joints in the planar ground that is the plane XY of the coordinate system and the joints on a lateral faces are also attached to fixed joints in the lateral wall that is the plane ZY of the coordinate system (Fig. 2).

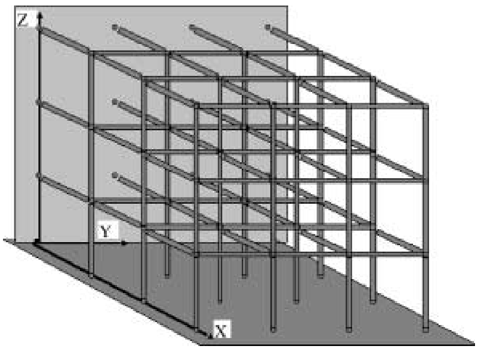

Fig. 2 $3 \times 3 \times 3$ annex building

We solve the annex building problem in this paper. This result is useful from an algorithmic point of view, for instance the number of the joints is $O(k\ell m)$ in the case of the annex building, and the size of the auxiliary graph of the framework, and hence the time complexity of the proposed algorithm, is $O(km)$, while according to Maxwell (1864) the time complexity would be $O[(k\ell m)^{2.4}]$. Our result will generalize this for higher dimensions and for stairs.

2.2 Motion

Consider a cube framework. This framework has small motion if any four joints of the cube are nonplanar during the motion of the joint if they were nonplanar in the original cube. A cube grid framework has small motion if each of its cubes has small motion.

In the annex building the intersection of the base face and the lateral face is called the base line.

Lemma 2.1. If a rod is originally parallel with the base line then it is also parallel with the base line during any small motion of the annex building.

Proof. It is sufficient to show that: if two opposite originally parallel rods of a cube framework that are not the same square faces of the cube are parallel to each other during any small motion of the joints, then the other two originally parallel rods will be parallel with the former two during any small motion of the joints.

$ABCDEFGH$ are the joints of the cube framework. The rod AB is parallel rod HG. The joint E can move on a circle and similarly F. Let E move to E_1 and let F move to F_1. If E_1 is fixed then there are two places for F_1 on the circle of F, but only one satisfies the condition of the small motion. In this case EF is parallel with AB or HG and DC also.

We can see a counter-example for the necessity of the small motion on the right-hand side of Fig. 3. If rod E_1F_1 is in the plane $ABGH$ then E_1 can move to E_2 and F_1 can move to F_2. Hence E_2 and F_2 are on different sides of the plane $ABGH$, therefore rod E_2F_2 is not parallel to AB or HG.

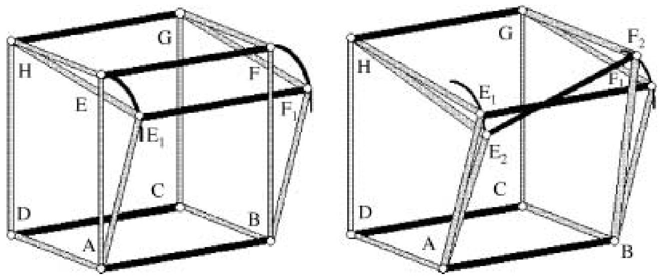

Fig. 3 Small motion on the left side, no small motion on the right side

Denote by $\ell_{i,j}$ the set of the rods obtained by a translation of the base line by i units of direction X and j units of direction Z. In the case of small motion each rod in $\ell_{1,1}$ is parallel to $\ell_{0,1}$, $\ell_{1,0}$ and $\ell_{0,0}$ or the base line as a consequence of Lemma 2.1.; $\ell_{i,j}$ is parallel to $\ell_{i-1,j}$, $\ell_{i,j-1}$ and the base line using mathematical induction. Hence each rod that originally is parallel to the base line, is also parallel to the base line during any small motion of the annex building.

Using this lemma we give a theorem for the rigidity of an annex building. Denote by $y_{i,j}$ the distance of $\ell_{i,j}$ from the XY plane. Naturally $y_{0,1}$, $y_{1,0}$ and $y_{0,0} = 0$. Denote by $h_{i,j}$ the face diagonal brace between $\ell_{i,j}$ and $\ell_{i-1,j}$, and similarly by $v_{i,j}$ the face diagonal brace between $\ell_{i,j}$ and $\ell_{i,j-1}$. However the place of $h_{i,j}$ in the strip between $\ell_{i,j}$ and $\ell_{i-1,j}$ is not important. If there is such a face diagonal brace $h_{i,j}$ anywhere in the strip between $\ell_{i,j}$ and $\ell_{i-1,j}$ then $y_{i,j} = y_{i-1,j}$. Similarly if there is a face diagonal brace $v_{i,j}$ the face diagonal brace between $\ell_{i,j}$ and $\ell_{i,j-1}$ then $y_{i,j} = y_{i,j-1}$.

2.3 Bracing graph

Let us give the definition of the auxiliary graph or bracing graph (Fig. 4).

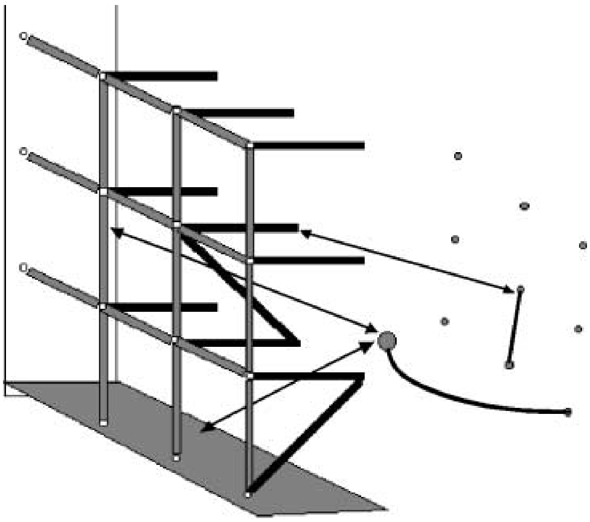

Fig. 4 Bracing graph of the annex building

Definition 2.2. $B(P_{i,j}, E)$ bracing graph. Let $\ell_{i,j}$ correspond to points $P_{i,j}$ in the bracing graph if $i \neq 0$ and $j \neq 0$. If i or j equal to 0 then $\ell_{i,j}$ correspond to the same point $P_{0,0}$; $(P_{i-1,j}, P_{i,j}) \in E$ i.e. there is an edge between $P_{i-1,j}$ and $P_{i,j}$ if and only if there is a $h_{i,j}$ face diagonal brace, and $(P_{i,j-1}, P_{i,j}) \in E$, i.e. there is an edge between $P_{i,j-1}$ and $P_{i,j}$ if and only if there is a $v_{i,j}$ face diagonal brace.

2.4 The main result

We give a necessary and sufficient condition for the infinitesimal rigidity of an annex building.

Theorem 2.2. The $k \times \ell \times m$ annex building is infinitesimal rigid if and only if its bracing graph is connected.
Proof. If there is a $h_{i,j}$ face diagonal brace in its strip then $y_{i,j} = y_{i-1,j}$, and also if there is a $v_{i,j}$ the face diagonal brace its strip then $y_{i,j} = y_{i,j-1}$. If the bracing graph is connected then we can get from $P_{0,0}$ to every $P_{i,j}$ hence $y_{i,j} = 0$ for any i, j, therefore the annex building is infinitesimally rigid because there is no infinitesimal motion in direction X or in direction Z.

If the bracing graph is not connected then there exists a component of the bracing graph which does not contain $P_{0,0}$. The rods in the corresponding $\ell_{i,j}$ of this component can have a (possibly infinitesimal) motion in the direction of the base line.

Corollary 2.1. The number of the braces is independent of ℓ. We need km diagonal braces for the infinitesimal rigidity.

Corollary 2.2. Using this theorem we can test the rigidity of an annex building in linear steps because the step of the test of the graph connectivity is $O(km)$, this is independent of ℓ.

3 Stairs

3.1 Stairs in the three-dimensional space

Definition 3.1. A stair is a framework obtained from the annex building by deleting some cube as follows. The coordinate of a cube (i, g, j) is defined as the coordinate of the nearest point of the cube to the origin. A cube with coordinate (i, g, j) can be deleted from the grid only if either all the cubes with coordinates (i_1, g_1, j_1), $i \leq i_1$, $g \leq g_1$, $j \leq j_1$ are also deleted, or if all the cubes with coordinates (i_1, g_1, j_1), $i \leq i_1$, $g \geq g_1$, $j \leq j_1$ are also deleted. The stairs are shown in Fig. 5.

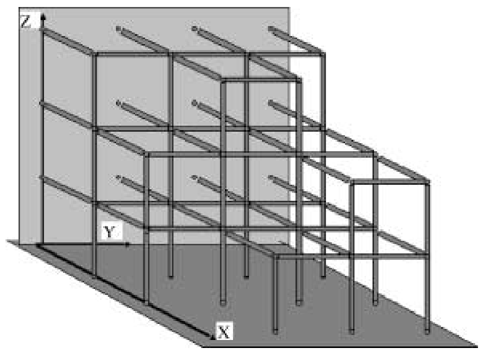

Fig. 5 Stair

By this definition any row of cubes in the stair contain a sequence of adjacent cubes only.

Similarly to the bracing graph of annex building, we can define the bracing graph of the stairs.

Theorem 3.1. The stair is infinitesimally rigid if and only if its bracing graph is connected.
Proof. Similar to the proof of Theorem 2.2.

3.2 Stairs in higher dimensional space

The higher dimensional annex building could be defined from a higher dimensional cube grid framework like in the three-dimensional case. The higher dimensional annex building is a higher dimensional cube grid framework and its joints on each neighbouring hyperface except one are attached to fixed joints in the coordinate hyperfaces.

We can get to a higher dimensional stair if we delete some cubes from the higher dimensional cube grid like in the three-dimensional case.

Similarly we can define the bracing graph of higher dimensional stairs and we can obtain a theorem similar to Theorem 3.1.

Theorem 3.2. The higher dimensional stairs are infinitesimally rigid if and only if their bracing graph is connected.

Proof. Similar to the proof of Theorem 2.2.

Remark 3.1. The annex building might be rigid but not infinitesimally, if we use long braces to fix the joint, that is opposite to the origin, to the coordinate hyper faces.

Corollary 3.2. In the case of dimension 2 we obtain the trivial result that for the rigidity of a two-dimensional annex building one needs to insert diagonal braces into every row of the square grid.

Acknowledgements The author is grateful for the help of András Recski. This research was supported by the Foundation of Hungarian Science and by the National Science Foundation (grant number: OTKA T 017181).

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